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ON THE CONSISTENCY AND EQUIVALENCE OF CERTAIN DEFINITIONS OF SUMMABILITY*

BY

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1. Introduction

Many definitions have been proposed for the value of a divergent series, those of Cesàro and Hölder being familiar examples. All of the definitions proposed are generalizations of convergence;† that is, they evaluate any convergent series to the value to which it converges. Thus all these definitions give the same value to a convergent series. The fundamental question as to whether, when each of two definitions gives a value to a divergent series, the two values are the same, seems as yet to have received no attention.‡ That two definitions, both generalizations of convergence, may give different values to the same divergent series, is seen by the following example. Let the sequence defining the series be $x_n = (-1)^{n+1} \log n \quad (n = 1, 2, \cdots)$; and let the value of the sequence be defined in two different ways by the limits of the sequences (y_n), (z_n), where

$$y_n = \frac{1}{n}\sum_{k=1}^{n} x_k, \qquad z_n = \frac{1}{n}\sum_{k=2}^{n}\left[1 + \frac{(-1)^{k+1}}{\log k}\right]x_k \quad (n = 2, 3, \cdots).$$

It can easily be verified that each of the definitions is a generalization of convergence; whereas

$$\lim_{n=\infty} y_n = 0, \qquad \lim_{n=\infty} z_n = 1.$$

Furthermore the sequence (x_n) may be made to give any preassigned value λ whatever if we choose as the definition for the value of (x_n) the limit of s_n, where $s_n = (1-\lambda)y_n + \lambda z_n$. It is accordingly a matter of the first importance to know under what circumstances two definitions are *consistent*; that is, under what circumstances we have a right to assert that whenever each of two definitions gives a value to a sequence, the two values are the same.

* Presented to the Society, September 8, 1914.

† Such definitions are sometimes said to satisfy the condition of consistency; the word consistency is used in this paper in a different sense.

‡ Of course consistency is self-evident in the trivial case in which one definition evaluates *all* series evaluated by another definition, giving the same values.

The principal result of this paper is that all definitions of summability of a certain class are consistent.

Another result of some interest is the determination of a criterion for the equivalence of two definitions of summability; two definitions being defined as *equivalent* when each evaluates to the value ξ every sequence evaluated by the other to the value ξ. The interest in this idea seems hitherto to have been directed to the proof of the equivalence of Cesàro's and Hölder's definitions for the same order of summability, though other similar special questions have been considered. In this paper a criterion is established for the equivalence of any two definitions of a certain class, from which criterion the equivalence of Cesàro's and Hölder's definitions follows as a very special case.

Other points considered are: the specification of a definition which shall evaluate the sum of two given sequences summable by two stated definitions; the establishment of a necessary condition for summability, analogous to the well-known conditions for the cases of convergence and of Hölder-summability; and the permissibility of omitting or adjoining an element at the beginning of a summable sequence without altering either its summability or the value to which it is summable.

We shall be concerned with a special type* of definition of summability. Let $(x_n) = x_1, x_2, \cdots$ be a sequence, and

$$(a_{n,k}) = \left\| \begin{matrix} a_{1,1}, & 0, & 0, & \cdots \\ a_{2,1}, & a_{2,2}, & 0, & \cdots \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & \cdots \\ \vdots & \vdots & \vdots & \end{matrix} \right\|$$

an infinite matrix of numbers, real or complex. Then the transformation

$$y_n = \sum_{k=1}^{n} a_{n,k} x_k \qquad (n = 1, 2, \cdots)$$

defines the sequence $(y_n) = y_1, y_2, \cdots$. We indicate the transformation or the matrix by A and the relation by $(y_n) = A(x_n)$. If

$$\lim_{n=\infty} y_n = \xi,$$

we define ξ to be the value given to (x_n) by the definition or transformation A.

If $a_{nn} \neq 0$ $(n = 1, 2, \cdots)$, we may solve for x_n,

$$x_n = \sum_{k=1}^{n} \alpha_{n,k} y_k \qquad (n = 1, 2, \cdots);$$

* This type was first studied by Silverman, Missouri dissertation, 1910, and *University of Missouri Studies, Mathematics Series*, vol. 1, No. 1 (1913); Toeplitz, Prace matematyczno fizyczno, vol. 22 (1911), p. 113; Smail, Columbia dissertation, 1913; and Schur, Mathematische Annalen, vol. 74 (1913), p. 447.

we denote this transformation by A^{-1} and the relation by $(x_n) = A^{-1}(y_n)$. If $(y_n) = A(x_n)$ and $z_n = B(y_n)$, then $z_n = B(A(x_n))$, the transformation being BA. If $AB = BA$, A and B are *permutable*. If A and B correspond to $(a_{n,k})$ and $(b_{n,k})$ respectively, $\alpha A + \beta B$ will correspond to $(\alpha a_{n,k} + \beta b_{n,k})$. If $A_1, A_2, \cdots$ correspond to $(a_{n,k}^{(1)}), (a_{n,k}^{(2)}), \cdots$ respectively, $\alpha_1 A_1 + \alpha_2 A_2 + \cdots$ will correspond to $(a_{n,k})$ if

$$\lim_{p=\infty} [\alpha_1 a_{n,k}^{(1)} + \alpha_2 a_{n,k}^{(2)} + \cdots + \alpha_p a_{n,k}^{(p)}]$$

exists and equals $a_{n,k}$. If $(y_n) = A(x_n)$, and (y_n) has the limit ξ whenever (x_n) has the limit ξ, A is *regular*. It will be seen that if A and B are regular, then AB, and for any constant α, $\alpha A + (1 - \alpha)B$ are regular.

A necessary and sufficient condition that A be regular* is

$$(1) \quad (a) \quad \lim_{n=\infty} a_{n,k} = 0, \qquad (b) \quad \lim_{n=\infty} \sum_{k=1}^{n} a_{n,k} = 1, \qquad (c) \quad \sum_{k=1}^{n} |a_{n,k}| < K.$$

The simplest examples of regular transformations are

$$E: \quad a_{n,k} = 0, \qquad n \neq k; \qquad a_{n,n} = 1;$$

and

$$M: \quad a_{n,k} = \frac{1}{n}.$$

Schur has studied the transformation $\alpha E + (1 - \alpha)M$. It is natural to consider the more general transformation

$$\alpha_0 E + \alpha_1 M + \alpha_2 M^2 + \cdots + \alpha_n M^n,$$

or still more generally the symbol

$$\alpha_0 E + \alpha_1 M + \alpha_2 M^2 + \cdots,$$

and to ask under what conditions it defines a regular transformation.

THEOREM I. *If $f(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots$ is analytic within and on the boundary of a circle of unit radius about the origin, and $f(1) = 1$, then the symbol $\alpha_0 E + \alpha_1 M + \alpha_2 M^2 + \cdots$ gives rise to a regular transformation.*

Let $E, M, M^2, \cdots$ correspond to matrices $m_{n,k}^{(0)}, m_{n,k}^{(1)}, m_{n,k}^{(2)}, \cdots$, respectively. Then we have

$$(a) \qquad \sum_{r=0}^{p} |\alpha_r m_{n,k}^{(r)}| \leqq \sum_{r=0}^{p} |\alpha_r|,$$

* The sufficiency of the condition was proved by Silverman and the necessity by Toeplitz in the papers previously cited. We shall use only the fact of sufficiency; indeed certain results obtained by Schur by means of the necessity of the condition are here proved without this part of the criterion.

since $0 \leqq m_{n,k}^{(r)} \leqq 1$. Hence

$$\lim_{p=\infty} \sum_{r=0}^{p} \alpha_r m_{n,k}^{(r)}$$

exists uniformly in k; call its value $a_{n,k}$. Therefore*

$$\lim_{n=\infty} a_{n,k} = \lim_{n=\infty} \lim_{p=\infty} \sum_{r=0}^{p} \alpha_r m_{n,k}^{(r)} = \lim_{p=\infty} \lim_{n=\infty} \sum_{r=0}^{p} \alpha_r m_{n,k}^{(r)} = 0.$$

$$\text{(b)} \qquad \sum_{k=1}^{n} a_{n,k} = \sum_{k=1}^{n} \lim_{p=\infty} \sum_{r=0}^{p} \alpha_r m_{n,k}^{(r)} = \lim_{p=\infty} \sum_{r=0}^{p} \alpha_r \sum_{k=1}^{n} m_{n,k}^{(r)}$$

$$= \lim_{p=\infty} \sum_{r=0}^{p} \alpha_r = f(1) = 1.$$

$$\text{(c)} \qquad \sum_{k=1}^{n} |a_{n,k}| \leqq \sum_{r=0}^{\infty} |\alpha_r|,$$

since, from the hypothesis, the series for $f(z)$ converges absolutely for $z = 1$. As the conditions (a), (b), (c) of (1) are satisfied, the theorem is proved.

COROLLARY. *The numbers $a_{n,k}$ are given in terms of $f(z)$ by the formula*

$$\text{(2)} \qquad a_{n,k} = \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-1)!}{(n-h)!(h-k)!(k-1)!} f\left(\frac{1}{h}\right).$$

We prove this first when $f(z) = z^r$, so that the transformation defined is M^r. It is to be shown that

$$\text{(3)} \qquad m_{n,k}^{(r)} = \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-1)!}{(n-h)!(h-k)!(k-1)!} \frac{1}{h^r}.$$

Suppose this holds for any r; to see that it holds for $r+1$, we write

$$\begin{aligned} m_{n,k}^{(r+1)} &= \frac{1}{n} \sum_{q=k}^{n} m_{q,k}^{(r)} \\ &= \frac{1}{n} \sum_{q=k}^{n} \sum_{h=k}^{q} (-1)^{h-k} \frac{(q-1)!}{(q-h)!(h-k)!(k-1)!} \frac{1}{h^r} \\ &= \frac{1}{n} \sum_{h=k}^{n} \sum_{q=h}^{n} (-1)^{h-k} \frac{(q-1)!}{(q-h)!(h-k)!(k-1)!} \frac{1}{h^r} \\ &= \frac{1}{n} \sum_{h=k}^{n} \frac{(-1)^{h-k}}{(h-k)!(k-1)!} \frac{1}{h^r} \sum_{q=h}^{n} \frac{(q-1)!}{(q-h)!} \\ &= \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-1)!}{(n-h)!(h-k)!(k-1)!} \frac{1}{h^{r+1}}. \end{aligned}$$

* Osgood, *Lehrbuch der Funktionentheorie*, vol. 1, 2d ed., p. 593.

As evidently (3) is true for $r = 0$, it is true for all values of r. Finally, to prove (2) in general, multiply (3) by α_r and sum from $r = 0$ to $r = \infty$. Then

$$\sum_{r=0}^{\infty} \alpha_r\, m_{n,k}^{(r)} = \sum_{r=0}^{\infty} \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-1)!}{(n-h)!(h-k)!(k-1)!} \frac{\alpha_r}{h^r}$$

$$= \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-1)!}{(n-h)!(h-k)!(k-1)!} \sum_{r=0}^{\infty} \frac{\alpha_r}{h^r},$$

which agrees with (2).

2. Consistency of transformations permutable with M

We shall now determine a sufficient condition for the consistency of two regular definitions of summability. We shall refer to any transformation of the form

$$y_n = f_n\, x_n \tag{4}$$

as a *multiplication*. We have occasion to use Euler's transformation*

$$y_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{(n-1)!}{(n-k)!(k-1)!} x_k, \tag{5}$$

which we shall call Δ. It satisfies the condition

$$\Delta^2 = E, \tag{6}$$

as may be seen by actual repetition of (5) or more easily by solving the expanded formulae:

$$y_1 = x_1,$$
$$y_2 = x_1 - x_2,$$
$$y_3 = x_1 - 2x_2 + x_3,$$
$$\cdots\cdots\cdots$$

Lemma 1. *If M' denotes the multiplication*

$$y_n = \frac{1}{n} x_n, \tag{7}$$

then

$$M = \Delta M' \Delta, \qquad M' = \Delta M \Delta.$$

To prove this, form $M\Delta$; we have

$$y_n = \frac{1}{n} \sum_{k=1}^{n} \sum_{h=1}^{k} (-1)^{h-1} \frac{(k-1)!}{(k-h)!(h-1)!} x_h$$

* Bromwich, *An Introduction to the Theory of Infinite Series*, p. 303.

$$= \frac{1}{n}\sum_{h=1}^{n}(-1)^{h-1} x_h \left(\sum_{k=h}^{n}\frac{(k-1)!}{(k-h)!(h-1)!}\right)$$
$$= \frac{1}{n}\sum_{h=1}^{n}(-1)^{h-1} \frac{n!}{(n-h)!\,h!} x_h$$
$$= \sum_{h=1}^{n}(-1)^{h-1} \frac{(n-1)!}{(n-h)!(h-1)!}\left(\frac{x_h}{h}\right).$$

But this is obviously the result of applying Δ to the sequence (x_n/n),—that is, to the result of transforming (x_n) by M'. Hence

$$M\Delta = \Delta M'$$

and the proposition follows at once by (6).

LEMMA 2. *A necessary and sufficient condition that A be permutable with M is that A' be permutable with M', where $A' = \Delta A \Delta$.*

If $AM = MA$, then

$$A'M' = (\Delta A\Delta)(\Delta M\Delta) = \Delta(AM)\Delta = \Delta(MA)\Delta$$
$$= (\Delta M\Delta)(\Delta A\Delta) = M'A'.$$

Conversely, let $A'M' = M'A'$; then

$$AM = (\Delta A'\Delta)(\Delta M'\Delta) = \Delta(A'M')\Delta = \Delta(M'A')\Delta$$
$$= (\Delta M'\Delta)(\Delta A'\Delta) = MA.$$

LEMMA 3. *A necessary and sufficient condition that a transformation be permutable with M' is that it be a multiplication.*

Suppose the transformation A',

$$y_n = \sum_{k=1}^{n} a_{n,k}\, x_k \tag{8}$$

is permutable with M'. Then writing $M'A' = A'M'$, we have

$$\frac{1}{n}\sum_{k=1}^{n} a_{n,k}\, x_k = \sum_{k=1}^{n} a_{n,k}\frac{x_k}{k}.$$

Hence

$$\frac{a_{n,k}}{n} = \frac{a_{n,k}}{k},$$

or

$$(n-k)\,a_{n,k} = 0.$$

Therefore, when $k \neq n$, $a_{n,k} = 0$. Writing $a_{n,n} = f_n$, we have the multiplication (4).

Conversely, if we denote by A' the transformation (4), obviously

$$M'A' = A'M'.$$

Combining the two preceding lemmas, we have

LEMMA 4. *A necessary and sufficient condition that A be permutable with M is that there exist a multiplication A' such that $A = \Delta A' \Delta$.*

We shall now say that *A corresponds to A'* if $A = \Delta A' \Delta$.

LEMMA 5. *If A and B correspond to A' and B' respectively, then AB corresponds to $A'B'$.*

For $AB = (\Delta A' \Delta)(\Delta B' \Delta) = \Delta(A'B')\Delta$.

LEMMA 6. *Two transformations, each permutable with M, are permutable with each other.*

For two multiplications are obviously permutable; hence if A, B correspond respectively to A', B', then AB and BA, which correspond by Lemma 5 to $A'B'$ and $B'A'$, are equal.

THEOREM II. *All regular definitions permutable with M are consistent.*

Let A and B be any two regular definitions permutable with M, evaluating the sequence (x_n) to ξ and η respectively. Then (x_n) is evaluated by BA to ξ and by AB to η. Since by Lemma 6, $AB = BA$, it follows that $\xi = \eta$.

THEOREM III. *If A, B are regular definitions permutable with M, and if A evaluates (x_n) to ξ and B evaluates (y_n) to η, then AB evaluates $(x_n + y_n)$ to $\xi + \eta$.*

For evidently AB evaluates (x_n) to ξ and (y_n) to η.

THEOREM IV. *A necessary and sufficient condition that A be permutable with M is that there exist numbers $f_1, f_2, \cdots$ such that*

$$a_{n,k} = \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-1)!}{(n-h)!(h-k)!(k-1)!} f_h. \tag{9}$$

If A is permutable with M, then by Lemma 4, $A = \Delta A' \Delta$, where A' is given by (4). Hence we have for A the formula

$$\begin{aligned} y_n &= \sum_{h=1}^{n} (-1)^{h-1} \frac{(n-1)!}{(n-h)!(h-1)!} f_h \\ &\qquad \times \left(\sum_{k=1}^{h} (-1)^{k-1} \frac{(h-1)!}{(h-k)!(k-1)!} x_k \right) \\ &= \sum_{k=1}^{n} \left(\sum_{h=k}^{n} (-1)^{h-k} \frac{(n-1)!}{(n-h)!(h-k)!(k-1)!} f_h \right) x_k, \end{aligned} \tag{10}$$

which proves (9).

Conversely, if (9) is satisfied, (10) shows that $A = \Delta A' \Delta$, where A' is a multiplication.

3. REGULARITY OF TRANSFORMATIONS PERMUTABLE WITH M

We have seen that the symbol $\alpha_0 E + \alpha_1 M + \alpha_2 M^2 + \cdots$ gives rise to a regular transformation if $f(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots$ is analytic in the

unit circle. We consider next a general function $f(z)$ of the complex variable z; and we define $f(M)$, the corresponding function of M, to be a transformation for which the numbers $a_{n,k}$ are given by the formula*

$$a_{n,k} = \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-1)!}{(n-h)!(h-k)!(k-1)!} f\left(\frac{1}{h}\right). \tag{11}$$

It is evident from Theorem IV of the preceding section that $f(M)$ is permutable with M, and from Lemma 5, that if $h(z) = f(z)g(z)$, then $h(M) = f(M)g(M)$. We shall now prove a theorem which contains Theorem I as a special case.

THEOREM V. *The transformation $f(M)$ is regular if $f(z)$ is analytic within and on the boundary of the circle C of radius $\frac{1}{2}$ about the point $\frac{1}{2}$, and $f(1) = 1$.*

By hypothesis $f(z)$ is analytic in a circle C_ϵ of radius $\frac{1}{2} + \epsilon$ about the point $\frac{1}{2}$, where ϵ is a sufficiently small positive number. Since all the points $1, \frac{1}{2}, \frac{1}{3}, \cdots$ lie inside C_ϵ, we have by Cauchy's integral-theorem

$$f\left(\frac{1}{h}\right) = \frac{1}{2\pi i} \int_{C_\epsilon} \frac{f(t)}{t - \frac{1}{h}}\, dt.$$

Substituting this value in (11),

$$a_{n,k} = \frac{1}{2\pi i} \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-1)!}{(n-h)!(h-k)!(k-1)!} \int_{C_\epsilon} \frac{f(t)}{t - \frac{1}{h}}\, dt$$

$$= \frac{1}{2\pi i} \int_{C_\epsilon} \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-1)!}{(n-h)!(h-k)!(k-1)!} \frac{f(t)}{t - \frac{1}{h}}\, dt.$$

But†

$$\sum_{h=k}^{n} (-1)^{h-k} \frac{n!}{(n-h)!(h-k)!(k-1)!} \frac{1}{t - \frac{1}{h}} = t^{n-k-1} \prod_{h=k}^{n} \frac{1}{t - \frac{1}{h}}, \qquad n > k,$$

so that

$$a_{n,k} = \frac{1}{2\pi i n} \int_{C_\epsilon} f(t)\, t^{n-k-1} \prod_{h=k}^{n} \frac{1}{t - \frac{1}{h}}\, dt$$

$$= \frac{1}{2\pi i n} \int_{C_\epsilon} \frac{f(t)}{t^2} \prod_{h=k}^{n} \frac{1}{1 - \frac{1}{ht}}\, dt, \qquad n > k.$$

* This formula is identical with (2) of Section 1.

† This may be proved at once by resolving the right-hand side into partial fractions.

Now C_ϵ is a circle whose diameter joins the points $-\epsilon$ and $1+\epsilon$. Thus when t lies on C_ϵ, ht lies on a circle whose diameter joins the points $-h\epsilon$ and $h(1+\epsilon)$, and $1/ht$ lies on a circle whose diameter joins the points $-1/h\epsilon$ and $1/h(1+\epsilon)$; hence

$$\left|1-\frac{1}{ht}\right| \geqq 1-\frac{1}{h(1+\epsilon)}.$$

Let L be a constant greater than the absolute value, on C_ϵ, of $f(t)/t^2$, and let $\delta = 1/(1+\epsilon)$, and $N = L(\frac{1}{2}+\epsilon)$. Then

$$|a_{n,k}| \leqq \frac{L}{2\pi n}\int_{C_\epsilon}\prod_{h=k}^{n}\frac{1}{1-\dfrac{\delta}{h}}\,ds = \frac{N}{n}\prod_{h=k}^{n}\frac{h}{h-\delta}, \qquad n>k.$$

To prove that $f(M)$ is regular we shall show that the conditions (1) are fulfilled.

(*a*) We find

$$|a_{n,k}| \leqq \frac{N}{n}\prod_{h=k}^{n}\frac{h}{h-\delta}$$
$$= \frac{(n-k)^{k-\delta}\dfrac{1\cdot 2\cdot\cdots\cdot(n-k)}{(k-\delta)(k+1-\delta)\cdots(n-\delta)}}{(n-k)^{k}\dfrac{1\cdot 2\cdot\cdots\cdot(n-k)}{k(k+1)\cdot\cdots\cdot n}}\, N\left(\frac{n-k}{n}\right)^{\delta}\frac{1}{n^{1-\delta}}.$$

Hence

$$\lim_{n=\infty}|a_{n,k}| = \frac{\Gamma(k-\delta)}{\Gamma(k)}\cdot N\cdot 1\cdot 0 = 0.$$

(*b*) We have

$$\sum_{k=1}^{n} a_{n,k} = \sum_{k=1}^{n}\sum_{h=k}^{n}(-1)^{h-k}\frac{(n-1)!}{(n-h)!(h-k)!(k-1)!}f\left(\frac{1}{h}\right)$$
$$= \sum_{h=1}^{n}\sum_{k=1}^{h}(-1)^{h-k}\frac{(n-1)!}{(n-h)!(h-k)!(k-1)!}f\left(\frac{1}{h}\right).$$

Now the coefficient of $f(1/h)$ may be written

$$\frac{(n-1)!}{(n-h)!(h-1)!}\sum_{k=1}^{h}(-1)^{h-k}\frac{(h-1)!}{(h-k)!(k-1)!},$$

which equals* unity for $h=1$ and zero for all other values of h. Hence

$$\sum_{k=1}^{n} a_{n,k} = f(1) = 1.$$

* The summation is obviously equal to the sum of the coefficients in the expansion of $(-1+x)^{h-1}$.

(*c*) We first find

$$|a_{n,1}| + |a_{n,2}| \leqq \frac{N}{n}\prod_{h=1}^{n}\frac{h}{h-\delta} + \frac{N}{n}\prod_{h=2}^{n}\frac{h}{h-\delta}$$

$$= \frac{N}{n}\,\frac{2}{1-\delta}\prod_{h=3}^{n}\frac{h}{h-\delta},$$

and then, by mathematical induction,

$$\sum_{k=1}^{p}|a_{n,k}| \leqq \frac{Np}{n(1-\delta)}\prod_{h=p+1}^{n}\frac{h}{h-\delta} \qquad (p<n),$$

so that

$$\sum_{k=1}^{n-1}|a_{n,k}| \leqq \frac{N(n-1)}{n(1-\delta)}.$$

Since the limit, as n becomes infinite, of the expression on the right exists, it follows that there exists a positive constant H such that

$$\sum_{k=1}^{n-1}|a_{n,k}| < H.$$

Furthermore, since $a_{n,n} = f(1/n)$, there exists a positive constant J such that

$$|a_{n,n}| < J.$$

Hence

$$\sum_{k=1}^{n}|a_{n,k}| < H + J.$$

COROLLARY. *The Hölder and Cesàro means of order r are respectively $f(M)$ and $g(M)$, where*

$$f(z) = z^r, \tag{12}$$

$$g(z) = \frac{r!\,z^r}{(1+z)(1+2z)\cdots(1+\overline{r-1}z)}. \tag{13}$$

The proof is immediate. It may at once be verified that the Hölder and Cesàro means are permutable with M; it is therefore sufficient to give for each case a function analytic in C, having the value 1 for $z = 1$ and the value $a_{n,n}$ for $1/n$. For the Hölder mean of order r, $a_{n,n} = 1/n^r$; hence the transformation is defined by $f(z) = z^r$. For the Cesàro mean of order r,

$$a_{n,n} = \frac{r!}{(r+n-1)(r+n-2)\cdots n}$$

$$= \frac{r!}{n^r\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{r-1}{n}\right)};$$

so that the transformation is defined by the function $g(z)$ given above.

4. A case of irregularity

As a case of irregularity it will be of value to study the effect of poles of the function $f(z)$. The simplest function possessing a pole of the first order and having the value 1 at $z = 1$ is $f(z) = (1-\rho)/(z-\rho)$, where $\rho \neq 1$.

Lemma 1. *The function*

$$f(z) = \frac{1-\rho}{z-\rho},$$

where $\rho \neq 1$ is a point within or on the boundary of the circle C, does not define a regular transformation.

Disregarding the cases in which ρ is the reciprocal of a positive integer, since in those cases the formula (11) for the coefficients of the matrix corresponding to $f(z)$ breaks down, and excluding the case $\rho = 0$, since in that case $f(z)$ defines the transformation M^{-1}, which is obviously not regular, we proceed to set up a sequence (x_n) and the transformed sequence (y_n) in such a way that the former has the limit zero, while the latter does not. It will be simpler to define (y_n) first. We may then find (x_n) by performing the transformation corresponding to $1/f(z) = (z-\rho)/(1-\rho)$, so that

$$(1-\rho)x_n = \frac{1}{n}\sum_{h=1}^{n} y_h - \rho y_n. \tag{14}$$

We take

$$y_n = \frac{\Gamma(n)}{\Gamma(n-p)}, \qquad p = \frac{1}{\rho} - 1. \tag{15}$$

The restrictions on ρ show that $\mathbf{R}(p) \geqq 0$ and that $p \neq 0, 1, 2, \cdots$. From these two facts we see that y_n is defined for $n = 1, 2, \cdots$; and that*

$$\lim_{n=\infty} y_n = \infty, \quad \text{if} \quad \mathbf{R}(p) > 0;$$
$$\lim_{n=\infty} |y_n| = 1, \quad \text{if} \quad \mathbf{R}(p) = 0;$$

so that in neither case does y_n approach zero.

On the other hand we find

$$\frac{\Gamma(n+1)}{\Gamma(n-p)} - \frac{\Gamma(n)}{\Gamma(n-p-1)} = (p+1)\frac{\Gamma(n)}{\Gamma(n-p)},$$

so that

$$y_n = \rho\frac{\Gamma(n+1)}{\Gamma(n-p)} - \rho\frac{\Gamma(n)}{\Gamma(n-p-1)},$$

* The ratio $\Gamma(n)/\Gamma(n-p)$ is readily studied by Stirling's Theorem. It is seen that the limit is 0 or ∞ according as $\mathbf{R}(p) <$ or > 0; if $\mathbf{R}(p) = 0$, the absolute value of the ratio has the limit 1.

and

$$\sum_{h=1}^{n} y_h = \rho \frac{\Gamma(n+1)}{\Gamma(n-p)} - \rho \frac{1}{\Gamma(-p)}. \tag{16}$$

Denoting by c the expression $\rho/\Gamma(-p)$, which is independent of n, we have from (14), (15), and (16),

$$(1-\rho) x_n = \rho \frac{\Gamma(n)}{\Gamma(n-p)} - \frac{c}{n} - \rho \frac{\Gamma(n)}{\Gamma(n-p)} = -\frac{c}{n},$$

so that

$$\lim_{n=\infty} x_n = 0.$$

LEMMA 2. *If $f(z)$ is analytic within and on the boundary of the circle C except for a single pole of the first order, and $f(1) = 1$, $f(z)$ does not define a regular transformation.*

If ρ is the pole, we must be able to write

$$f(z) = a \frac{1-\rho}{z-\rho} + g(z), \qquad a \neq 0, \tag{17}$$

where $g(z)$ is analytic throughout C. Assuming first $g(1) \neq 0$, we have from (17)

$$\begin{aligned} \frac{1-\rho}{z-\rho} &= \frac{1}{a} f(z) - \frac{1}{a} g(z) \\ &= \frac{1}{a} f(z) - \frac{g(1)}{a} \frac{g(z)}{g(1)}. \end{aligned} \tag{18}$$

Suppose now that $f(z)$ does define a regular transformation. Then calling A and B the regular transformations defined by $f(z)$ and $g(z)/g(1)$ respectively, we should find that the transformation

$$\frac{1}{a} A - \frac{g(1)}{a} B$$

is regular, since* from the assumption $f(1) = 1$ and from (18),

$$\frac{1}{a} - \frac{g(1)}{a} = 1.$$

Hence it would follow from (18) that the function $(1-\rho)/(z-\rho)$ defines a regular transformation, in contradiction to the preceding lemma.

In the case $g(1) = 0$, it follows from (17) that $a = 1$; and (18) becomes

$$\frac{1-\rho}{z-\rho} = f(z) - g(z),$$

---

* See p. 3.

so that

$$\frac{1}{2}\left(1+\frac{1-\rho}{z-\rho}\right)=\tfrac{1}{2}f(z)+\tfrac{1}{2}[1-g(z)].$$

If it is now assumed that $f(z)$ defines a regular transformation it will follow, since $1-g(z)$ defines a regular transformation, that

$$\phi(z)=\frac{1}{2}\left(1+\frac{1-\rho}{z-\rho}\right),$$

and hence that

$$\frac{1-\rho}{z-\rho}=2\phi(z)-1$$

defines a regular transformation. But this again contradicts the preceding lemma.

THEOREM VI. *If $f(z)$ has at least one pole, but is analytic except for poles within and on the boundary of the circle C, and $f(1)=1$, then $f(z)$ does not define a regular transformation.*

The number of poles must be finite; denote them, each repeated as often as its multiplicity indicates, by $\rho_1, \rho_2, \cdots, \rho_n$. Then

$$f(z)=\frac{1-\rho_1}{z-\rho_1}\cdot\frac{1-\rho_2}{z-\rho_2}\cdot\ldots\cdot\frac{1-\rho_n}{z-\rho_n}g(z),$$

where $g(z)$ is analytic throughout C. If $f(z)$ defines a regular transformation, so will the product of $f(z)$ by the analytic function

$$\frac{z-\rho_2}{1-\rho_2}\cdot\frac{z-\rho_3}{1-\rho_3}\cdot\ldots\cdot\frac{z-\rho_n}{1-\rho_n};$$

this product, however, possesses a single pole ρ_1 of the first order, and by Lemma 2 does not define a regular transformation.

5. ANALYTICALLY REGULAR TRANSFORMATIONS

We shall now use the term *analytically regular* to describe a transformation $f(M)$ defined by a function $f(z)$ analytic throughout C, having the value 1 for $z=1$. A number of properties of such transformations follow immediately from the results of the two preceding sections.

THEOREM VII. *All analytically regular definitions are consistent.*

This is evident by Theorem II, since all these transformations, being of the form (9), are, by Theorem IV, permutable with M.

THEOREM VIII. *If $f(M)$, $g(M)$ are analytically regular transformations, a necessary and sufficient condition that $f(M)$ should evaluate every sequence*

which $g(M)$ evaluates, giving it the same value, is that all the zeros of $g(z)$ in C should be zeros of at least as high order of $f(z)$.

If (x_n) be any sequence transformed by $f(M)$ to (u_n) and by $g(M)$ to (v_n), then (v_n) is transformed to (u_n) by $h(M)$, where $h(z) = f(z)/g(z)$. In order that the condition of the theorem be satisfied, it is necessary and sufficient that $h(M)$ be regular. As the only possible singularities of $h(z)$ are poles due to the zeros of $g(z)$, $h(M)$ will be regular if there are no poles (by § 3), that is, if the zeros of $g(z)$ are zeros of the same or higher orders of $f(z)$; and will not be regular in the contrary case (by § 4).

As an immediate deduction we have the two following theorems.

THEOREM IX. *If $f(M)$, $g(M)$ are analytically regular, a necessary and sufficient condition that they be equivalent is that $f(z)$, $g(z)$ have in C the same zeros with the same orders.*

COROLLARY. *The Hölder and Cesàro means of like order are equivalent.*

For the functions $f(z)$, $g(z)$ of (11), (12) are analytic in C; each has no zeros except $z = 0$, and this is in both cases a zero of order r; hence the two definitions are equivalent.

THEOREM X. *A necessary and sufficient condition that the analytically regular definition $f(M)$ be reversible (equivalent to convergence) is that $f(z)$ do not vanish in C.*

6. A NECESSARY CONDITION FOR SUMMABILITY

In the cases of convergence and of Cesàro and Hölder summability, there exists a simple form of necessary condition, applied usually to the general term of the infinite series, that is, to the difference of two elements of the sequence. We have a similar test in the case of a wide range of definitions of the type which we are considering.

THEOREM XI. *If the sequence (x_n) is transformed into a convergent sequence by the analytically regular transformation $f(M)$ defined by a function $f(z)$ which has no zeros within or on the boundary of the circle C' of radius $\frac{1}{4}$ about the point $\frac{1}{4}$, then*

$$\lim_{n=\infty} (x_n - x_{n-1}) = 0.$$

We suppose that the function $f(z)$ defines the transformation $f(M)$ which has as coefficients of its matrix $(f_{n,k})$, and that similarly $g(z)$, $a(z)$, $b(z)$ define respectively the transformations $g(M)$, $a(M)$, $b(M)$, with coefficients $(g_{n,k})$, $(a_{n,k})$, $(b_{n,k})$, where

$$g(z) = \frac{1}{f(z)}, \qquad a(z) = \frac{g\left(\dfrac{z}{z+1}\right)}{g(\frac{1}{2})}, \qquad b(z) = \frac{1}{z}a(z).$$

Since $f(z)$ has no zeros in C', $f[z/(z+1)]$ has no zeros in C, hence $a(z)$ is analytic in C and $a(M)$ is analytically regular. Now let

$$y_n = \sum_{k=1}^{n} f_{n,\,k}\, x_k;$$

then

$$x_n = \sum_{k=1}^{n} g_{n,\,k}\, y_k,$$

so that

$$x_n - x_{n-1} = g_{n,\,n}\, y_n + \sum_{k=1}^{n-1} (g_{n,\,k} - g_{n-1,\,k})\, y_k .$$

From

$$g_{n,\,k} = \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-1)!}{(n-h)!(h-k)!(k-1)!} g\left(\frac{1}{h}\right),$$

we find

$$g_{n,\,k} - g_{n-1,\,k} = \frac{g(\frac{1}{2})}{k-1} b_{n-1,\,k-1} \qquad k > 1,$$

$$g_{n,\,1} - g_{n-1,\,1} = -\, g(\tfrac{1}{2})\, a_{n-1,\,1},$$

$$g_{n,\,n} = \frac{g(\frac{1}{2})}{n-1} b_{n-1,\,n-1} .$$

Hence

$$x_n - x_{n-1} = -\, g(\tfrac{1}{2})\, a_{n-1,\,1}\, y_1 + g(\tfrac{1}{2}) \sum_{k=2}^{n} b_{n-1,\,k-1} \frac{y_k}{k-1}$$

$$= -\, g(\tfrac{1}{2})\, a_{n-1,\,1}\, y_1 + g(\tfrac{1}{2}) \sum_{k=1}^{n-1} b_{n-1,\,k} \frac{y_{k+1}}{k} .$$

The expression given by the summation sign is the result of applying the transformation $b(M)$ to the sequence

$$y_2,\ \frac{y_3}{2},\ \frac{y_4}{3},\ \cdots .$$

Since $b(z) = a(z)/z$, it suffices to apply to this sequence the transformation M^{-1}, and to the result the transformation $a(M)$. Hence the expression given by the summation is equal to

$$\sum_{k=1}^{n-1} a_{n-1,\,k}\, v_k ,$$

where

$$v_{n-1} = (n-1)\frac{y_n}{n-1} - (n-2)\frac{y_{n-1}}{n-2} = y_n - y_{n-1} .$$

That is,

$$
\begin{aligned}
x_n - x_{n-1} &= -g(\tfrac{1}{2})a_{n-1,1}y_1 + g(\tfrac{1}{2})\sum_{k=1}^{n-1} a_{n-1,k}(y_{k+1} - y_k) \\
&= g(\tfrac{1}{2})[a_{n-1,1}(y_2 - 2y_1) + a_{n-1,2}(y_3 - y_2) + \cdots \\
&\qquad + a_{n-1,n-1}(y_n - y_{n-1})].
\end{aligned} \tag{19}
$$

As the transformation $a(M)$ is analytically regular, and as, on account of the convergence of (y_n),

$$\lim_{n=\infty}(y_n - y_{n-1}) = 0,$$

it follows that

$$\lim_{n=\infty}(x_n - x_{n-1}) = 0,$$

as we wished to prove.

COROLLARY. *Under the same hypotheses as in the preceding theorem*

$$\lim_{n=\infty}\frac{x_n}{n} = 0.$$

As proof we have only to consider the Hölder means of the sequence x_1, $x_2 - x_1$, $x_3 - x_2$, $\cdots$.

We obtain a more general theorem by taking a function $f(z)$ which has a zero of order r at $z = 0$, but no other zeros in C'. Using the same notation as before, we have again (19). In the present case $z^r a(z)$ defines an analytically regular transformation; since the limit of the sequence $(y_n - y_{n-1})$ is zero, the result of applying to this sequence first $a(M)$, then M^r, must give a sequence whose limit is zero; therefore the sequence $x_n - x_{n-1}$ is evaluated to zero by M^r. The usual test for Hölder summability gives

$$\lim_{n=\infty}\frac{x_n - x_{n-1}}{n^r} = 0.$$

Thus we have

THEOREM XII. *If a sequence (x_n) is transformed into a convergent sequence by the analytically regular transformation $f(M)$ defined by a function $f(z)$ which has, except at $z = 0$, no zeros within or on the boundary of the circle C' of radius $\frac{1}{4}$ about the point $\frac{1}{4}$, then*

$$\lim_{n=\infty} f\left(\frac{1}{n}\right)(x_n - x_{n-1}) = 0.$$

7. Omission and adjunction of elements at the beginning of a sequence

It is natural to ask under what circumstances the evaluability of the sequence $x_1, x_2, x_3, \cdots$ by a definition insures the evaluability of the sequence $x_2, x_3,$

$x_4, \cdots$ by the same definition to the same value, and conversely. Two remarks should be made regarding this problem. In the first place, it is essentially equivalent to that of omitting or adjoining a term at the beginning of a series. To the series $u_1 + u_2 + u_3 + \cdots$ corresponds the sequence $x_1 = u_1$, $x_2 = u_1 + u_2$, $x_3 = u_1 + u_2 + u_3, \cdots$; to the series $u_2 + u_3 + u_4 + \cdots$ corresponds the sequence $x_2 - u_1,\ x_3 - u_1,\ x_4 - u_1, \cdots$, which differs, element by element, from the sequence $x_2, x_3, x_4, \cdots$, only by the convergent sequence $u_1, u_1, u_1, \cdots$. Secondly, it is clear that in the case of any regular definition the possibility of adjoining an element is independent of the value of the element adjoined; it is the mere fact of alteration in rank of the elements which affects the summability.

LEMMA. *If, by the rth Cesàro mean, the sequence $x_1, x_2, x_3, \cdots$ is transformed into $y_1, y_2, y_3, \cdots$, and the sequence $x_2, x_3, x_4, \cdots$ into $\eta_1, \eta_2, \eta_3, \cdots$, then*

$$y_n - \eta_n = \frac{r}{n} x_1 - \frac{n+r}{n}(y_{n+1} - y_n) - \frac{r}{n} y_n, \tag{20}$$

$$y_n - \eta_n = \frac{r}{n+r-1} x_1 - \frac{n-1}{n+r-1}(\eta_n - \eta_{n-1}) - \frac{r}{n+r-1}\eta_n. \tag{21}$$

From the formula for the Cesàro mean, we have

$$y_n = r\frac{(n-1)!}{(n+r-1)!}\sum_{k=1}^{n}\frac{(n+r-k-1)!}{(n-k)!}x_k,$$

$$\eta_{n-1} = r\frac{(n-2)!}{(n+r-2)!}\sum_{k=2}^{n}\frac{(n+r-k-1)!}{(n-k)!}x_k;$$

from these formulae follows at once

$$(n+r-1)y_n - (n-1)\eta_{n-1} = rx_1.$$

Solving for y_n, we obtain easily the second of the results to be proved; solving for η_{n-1} and replacing n by $n+1$, we obtain the first.

THEOREM XIII. *If a sequence (x_n) is transformed into a convergent sequence by the analytically regular transformation $f(M)$ defined by a function $f(z)$ which has, except at $z = 0$, no zeros within or on the boundary of the circle C', then the sequence obtained by omitting or adjoining an element at the beginning is transformed by $f(M)$ into a sequence converging to the same value.*

Suppose that $f(z)$ has at $z = 0$ a zero of order r, and write*

$$f(z) = \frac{r!\,z^r}{(1+z)(1+2z)\cdots(1+\overline{r-1}z)}g(z);$$

* The factor multiplied into $g(z)$ is exactly the function which defines the rth Cesàro mean given by (12).

then $g(z)$ is analytic in C and has no zeros in C'. Using the notation of the preceding lemma, it suffices to show that if either of the sequences (y_n), (η_n) can be evaluated by $g(M)$, the other will be evaluated to the same value. This will be shown by proving the stronger statement, that if either (y_n) or (η_n) is evaluable by $g(M)$, then

$$\lim_{n=\infty} (y_n - \eta_n) = 0.$$

If (y_n) is evaluable by $g(M)$, then by Theorem XI and its Corollary,

$$\lim_{n=\infty} (y_{n+1} - y_n) = 0, \qquad \lim_{n=\infty} \frac{y_n}{n} = 0;$$

hence by (20) the assertion follows. Similarly if (η_n) is evaluable by $g(M)$,

$$\lim_{n=\infty} (\eta_{n+1} - \eta_n) = 0, \qquad \lim_{n=\infty} \frac{\eta_n}{n} = 0,$$

which by (21) yields the result.

It will finally be shown that the condition on $f(z)$ is essentially necessary,—more accurately, that for any $\rho \neq 0$ in C' it is possible to construct a function $f(z)$ vanishing at ρ, and a sequence evaluable by $f(M)$ for which the dropping or adjoining an element is not permissible.

THEOREM XIV. *If $f(z) = (z - \rho)/(1 - \rho)$, where $\rho \neq 0$ is within or on the boundary of C', then there exists a sequence (x_n) which is transformed by $f(M)$ into a sequence converging to zero, and such that the sequences obtained by omitting and by adjoining an element are transformed by $f(M)$ into sequences which do not converge to zero.*

As in § 4, write $p = 1/\rho - 1$. Since ρ is in C', $\mathbf{R}(p) \geqq 1$; assuming first that $\rho \neq \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$, we find that $p \neq 1, 2, 3, \cdots$. Define*

$$x_n = \frac{\Gamma(n)}{\Gamma(n-p)};$$

then denoting the transformed sequence by (y_n),

$$y_n = \frac{1}{n\Gamma(1-p)},$$

so that

$$\lim_{n=\infty} y_n = 0.$$

If we apply $f(M)$ to the sequence $x_2, x_3, x_4, \cdots$, we have

$$\eta_n = \frac{1}{np^2\,\Gamma(-p)} + \frac{1}{p}\frac{\Gamma(n)}{\Gamma(n-p+1)};$$

* The sequences (x_n), (y_n) here are exactly the sequences (y_n), (x_n) respectively of § 4.

and applying $f(M)$ to the sequence $0, x_1, x_2, \cdots$,

$$\eta'_n = \frac{1}{n\Gamma(1-p)} - \frac{1}{p}\frac{\Gamma(n-1)}{n\Gamma(n-p-1)}.$$

It is evident that

$$\lim_{n=\infty} \eta_n = \infty$$

for $\mathbf{R}(p) > 1$, that is, for ρ inside C'; for ρ on the boundary of C', $\mathbf{R}(p) = 1$, and

$$\lim_{n=\infty} |\eta_n| = \frac{1}{|p|} \neq 0.$$

We see likewise, by writing

$$\frac{\Gamma(n-1)}{n\Gamma(n-p-1)} = \frac{n-2}{n}\frac{\Gamma(n-2)}{\Gamma(n-p-1)}$$

that

$$\lim_{n=\infty} \eta'_n = \infty$$

unless ρ is on the boundary of C'', and in this exceptional case,

$$\lim_{n=\infty} |\eta'_n| = \frac{1}{|p|} \neq 0.$$

We have excluded the cases $\rho = \frac{1}{2}, \frac{1}{3}, \cdots$. If ρ has any one of these values some of the earlier elements of the sequence (x_n) defined above become meaningless, since they involve in the denominators gamma functions of zero or negative integers; if, however, we replace each such meaningless element by zero, the preceding proof holds without alteration.

8. Conclusion

The class of analytically regular definitions considered in the preceding pages obviously includes a wide variety of definitions given by linear transformations. It does not, however, include all such definitions; for instance it fails to cover the logarithmic definitions of Riesz,* which are not permutable with M.

The consistency of all analytically regular definitions and the simplicity of the criteria for the equivalence and the relative generality of any two of them introduce a considerable degree of system into the study of such divergent series as may be successfully treated by this particular class of definitions. It is all the more important, therefore, to point out some desiderata in the theory. In the first place, some substitute for Theorem V, involving only

* Paris Comptes Rendus, vol. 149 (1909), p. 18.

real variables and conditions appropriate to real variables, is desirable, in order to remove the irksome requirement of analyticity in C. Again, consistency breaks down if the notion of limit be extended to include real one-signed infinity; for instance, the sequence $0, 1, 2, 3, \cdots$ is evaluated by the analytically regular definition* $2M - E$ to the value 0.

It is probable that a natural generalization exists of Cesàro's results† on the Cauchy-product of summable series, and of the theorem of Frobenius‡ on the behavior of a real power-series summable at an end of its interval of convergence.

Finally, the general results of the paper should admit of extension to the case of the limit of a continuous variable. The foundation for this extension exists in a paper by Silverman§ establishing conditions for regularity similar to (1). The further theorems analogous to those of the present article will be treated in a future paper.

* Other examples of the same type are contained incidentally in the proof of Theorem XIV.

† Bromwich, *An Introduction to the Theory of Infinite Series*, p. 315.

‡ Bromwich, l. c., p. 312.

§ These Transactions, vol. 17 (1916), p. 284; see also Bulletin of the American Mathematical Society, vol. 22 (1916), p. 459.

THE RESOLUTION INTO PARTIAL FRACTIONS OF THE RECIPROCAL OF AN ENTIRE FUNCTION OF GENUS ZERO*

BY

J. F. RITT

It will be desirable, in the paper which follows this one,† to have information as to the possibility of separating into partial fractions, the reciprocal of the entire function

$$\zeta(z) = \left(1 - \frac{z}{a_1}\right)^{p_1}\left(1 - \frac{z}{a_2}\right)^{p_2} \cdots \left(1 - \frac{z}{a_n}\right)^{p_n} \cdots,$$

where the exponents p_n are positive integers, and the zeros a_n any complex numbers except zero such that

$$\sum_{n=1}^{\infty} \frac{p_n}{|a_n|}$$

is convergent.

Proceeding as in the case of a polynomial, we might form the sum of the principal parts in the Laurent developments of $1/\zeta(z)$ at the poles a_n, and write

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty}\left[\frac{f_{n,1}}{1 - \frac{z}{a_n}} + \frac{f_{n,2}}{\left(1 - \frac{z}{a_n}\right)^2} + \cdots + \frac{f_{n,p_n}}{\left(1 - \frac{z}{a_n}\right)^{p_n}}\right],$$

where the expressions for the numbers f are readily found.

An example to be given later will show, however, that the series so found may be divergent. Thus, a discussion of the validity of the development is certainly in order.

Cauchy‡ discussed the resolution of meromorphic functions into simple elements, making suitable hypotheses for the behavior of the function on a sequence of closed curves.

Borel§ has considered the problem in the light of the more recent develop-

* Presented to the Society, April 29, 1916.

† The reader can go as far as the eighth article of the next paper without reading the present paper. We advise him, in fact, to do this.

‡ Cauchy, *Oeuvres Complètes*, 2d series, vol. 7, p. 324. For other references, see Lindelöf, *Calcul des Résidus*, Chapter II.

§ Borel, Annales de l'école normale, ser. 3, vol. 18 (1901); Acta Mathematica, vol. 24 (1901); also, *Fonctions Méromorphes*, Chapter IV.

ments in the theory of functions, and has extended Cauchy's results in several directions.

The case where the meromorphic function is the reciprocal of an entire function of genus zero does not appear to have received special notice, nor does it seem that a specialization of more general discussions would lead to the result of this paper.

We shall confine ourselves to the case where $\zeta(z)$ has only a finite number of multiple zeros. The theorem of this paper may then be stated:

If there exists an integer r such that, for $n \geqq r$,

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 + \frac{k}{n},$$

where $k > 2$, *the formal development of* $1/\zeta(z)$ *converges absolutely and uniformly to* $1/\zeta(z)$ *in every bounded domain in which* $1/\zeta(z)$ *is regular.**

If $\zeta(z)$ has no multiple zeros, the development to be considered is readily seen to be

$$\sum_{n=1}^{\infty} \frac{-1}{a_n \zeta'(a_n)} \frac{1}{1 - \dfrac{z}{a_n}},$$

where $\zeta'(z)$ is the first derivative of $\zeta(z)$.† Since $|a_n|$ goes to infinity with n, it is clear that this series will be absolutely and uniformly convergent in every bounded domain in which $1/\zeta(z)$ is regular, if

$$\sum_{n=1}^{\infty} \frac{1}{a_n \zeta'(a_n)}$$

is absolutely convergent. As to the case where there exist a finite number of multiple zeros, a similar statement is possible if we reject a finite number of terms of the development.

Let us examine now the development of the reciprocal of

$$\xi(z) = \frac{\sin \pi z^{\frac{1}{2}}}{\pi z^{\frac{1}{2}}} = \left(1 - \frac{z}{1^2}\right)\left(1 - \frac{z}{2^2}\right)\cdots\left(1 - \frac{z}{n^2}\right)\cdots.$$

Which value of $z^{\frac{1}{2}}$ we take is immaterial, provided that the same value is taken in both numerator and denominator. By direct calculation, the development for $1/\xi(z)$ is found to be

* If the condition of this theorem is satisfied, it will continue to be satisfied when equal zeros are written with distinct subscripts.

† Find the reciprocal of the first term in the development of $1/\zeta(z)$ for $z = a_n$.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{1-\dfrac{z}{n^2}},$$

which is divergent for every value of z.

Upon this example, we shall base our discussion of the general case. Writing equal roots now with distinct subscripts, we have

$$-a_n\zeta'(a_n)=\left(1-\frac{a_n}{a_1}\right)\cdots\left(1-\frac{a_n}{a_r}\right)\cdots\left(1-\frac{a_n}{a_{n-1}}\right)\left(1-\frac{a_n}{a_{n+1}}\right)\cdots.$$

Let us suppose that we can choose an integer r such that, for $n \geqq r$,

$$\left|\frac{a_{n+1}}{a_n}\right| > 1+\frac{k}{n},$$

where $k > 2$. Let h be any number less than k and greater than 2. Then, for n sufficiently great,

$$1+\frac{k}{n} > e^{h/n}.$$

Suppose r to have been chosen initially so that this last inequality holds for $n \geqq r$. Then, for $n \geqq r$,

$$\left|\frac{a_{n+1}}{a_n}\right| > e^{h/n}, \qquad \left|\frac{a_{n+2}}{a_{n+1}}\right| > e^{h/(n+1)}, \qquad \cdots, \qquad \left|\frac{a_{n+q}}{a_{n+q-1}}\right| > e^{h/(n+q-1)}.$$

Multiplying all these inequalities together,

$$\left|\frac{a_{n+q}}{a_n}\right| > e^{h\left(\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{n+q-1}\right)}$$
$$> e^{h\int_n^{n+q}\frac{dx}{x}} = e^{h[\log(n+q)-\log n]},$$

so that, finally,

$$\left|\frac{a_{n+q}}{a_n}\right| > \left(\frac{n+q}{n}\right)^h.$$

We have

$$-n^2\xi'(n^2)=\frac{(-1)^{n+1}}{2}$$
$$=\left(1-\frac{n^2}{1^2}\right)\cdots\left(1-\frac{n^2}{r^2}\right)\cdots\left(1-\frac{n^2}{\overline{n-1}^2}\right)\left(1-\frac{n^2}{\overline{n+1}^2}\right)\cdots.$$

Now, for $q > n \geqq r$,

$$\left|1-\frac{a_n}{a_q}\right| > 1-\left(\frac{n}{q}\right)^h > 1-\frac{n^2}{q^2},$$

so that

$$(1)\qquad \left|\prod_{q=n+1}^{\infty}\left(1-\frac{a_n}{a_q}\right)\right| > \prod_{q=n+1}^{\infty}\left(1-\frac{n^2}{q^2}\right).$$

Also, for $r \leqq q < n$,

$$\left|1-\frac{a_n}{a_q}\right| \geqq \left|\frac{a_n}{a_q}\right| - 1 > \left(\frac{n}{q}\right)^h - 1 > \left(\frac{n}{q}\right)^{h-2}\left(\frac{n^2}{q^2}-1\right).$$

Thus,

$$\left|\prod_{q=r}^{q=n-1}\left(1-\frac{a_n}{a_q}\right)\right| > \prod_{q=r}^{q=n-1}\left(\frac{n}{q}\right)^{h-2}\left(\frac{n^2}{q^2}-1\right)$$

$$> \left[\frac{(r-1)!\,n^{n-r}}{(n-1)!}\right]^{h-2}\left|\prod_{q=r}^{q=n-1}\left(1-\frac{n^2}{q^2}\right)\right|.$$

By Stirling's theorem,

$$(n-1)! = \sqrt{2\pi n}\, n^{n-1}\, e^{-n+\theta/12n},$$

where $0 < \theta < 1$, so that, for n sufficiently large,

$$(n-1)! < n^n e^{-n}.$$

Then, dropping the factor $(r-1)!$,

$$(2)\qquad \left|\prod_{q=r}^{q=n-1}\left(1-\frac{a_n}{a_q}\right)\right| > (n^{-r} e^n)^{h-2}\left|\prod_{q=r}^{q=n-1}\left(1-\frac{n^2}{q^2}\right)\right|.$$

Now, observing that for n sufficiently large, the first $r-1$ factors of

$$-\,a_n\,\zeta'(a_n)$$

are each greater in absolute value than unity, we have, from (1) and (2),

$$|a_n\,\zeta'(a_n)| > \left|\frac{(n^{-r}e^n)^{h-2}}{\left(1-\frac{n^2}{1^2}\right)\left(1-\frac{n^2}{2^2}\right)\cdots\left(1-\frac{n^2}{\overline{r-1}^2}\right)}\,n^2\,\xi'(n^2)\right|$$

$$> \frac{(n^{-r}e^n)^{h-2}}{2n^{2r-2}},$$

or finally,

$$|a_n\zeta'(a_n)| > \frac{n^{2-rh}\,e^{n(h-2)}}{2}.$$

From this last inequality, the absolute convergence of

$$\sum \frac{1}{a_n \zeta'(a_n)}$$

follows without difficulty, so that the development of $1/\zeta(z)$ is absolutely and uniformly convergent in every bounded domain in which $1/\zeta(z)$ is regular.

It is not difficult now to show that the series of fractions converges to $1/\zeta(z)$.* Let $\chi(z)$ be the entire function whose zeros are the moduli of the zeros of $\zeta(z)$. Then, ignoring the finite number of multiple zeros of $\chi(z)$,

$$\sum \frac{1}{|a_n \chi'(|a_n|)|}$$

is convergent. Let

$$\zeta_m(z) = \left(1 - \frac{z}{a_1}\right)\left(1 - \frac{z}{a_2}\right) \cdots \left(1 - \frac{z}{a_m}\right).$$

It is easily seen that for $n \leqq m$,

$$|a_n \zeta'_m(a_n)| \geqq |a_n \chi'(|a_n|)|.$$

Hence, the sum of the last q terms in the partial fraction development of $1/\zeta_{m+q}(z)$ goes to zero for every value of z as m and q go to infinity independently of each other. We can take m so as to make small at pleasure, for any particular value of z, the sum of the terms in the development of $1/\zeta(z)$ which involve zeros with subscripts greater than m. As q increases indefinitely, m remaining fixed, $1/\zeta_{m+q}(z)$ approaches $1/\zeta(z)$ and the coefficients in the development of $1/\zeta_{m+q}(z)$ approach those in the development of $1/\zeta(z)$. It follows readily that the development of $1/\zeta(z)$ converges to $1/\zeta(z)$.

Through considerations similar to those which precede, it can be shown that, if the zeros of $\zeta(z)$ are all real and positive, if an infinite number of them are of order one, and if, after arranging the zeros in order of magnitude (increasing), there exists an integer r such that, for $n \geqq r$,

$$\frac{a_{n+1}}{a_n} < 1 + \frac{k}{n},$$

where $0 < k < 2$, the partial fraction development of $1/\zeta(z)$ is divergent.

We have thus determined a value of the rate of increase of the moduli of the zeros of $\zeta(z)$ which is critical with respect to the convergence of the

* This fact is not important for the following paper.

development of $1/\zeta(z)$. It would be easy, however, to give examples where the zeros do not increase in absolute value as rapidly as we have supposed, and where the partial fraction development is convergent. For instance, if the zeros of $\zeta(z)$ are real and of alternating signs, weaker conditions will suffice to insure the convergence.*

* For example, the reciprocal of

$$\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^3}\right)$$

can be developed formally.

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ON A GENERAL CLASS OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS OF INFINITE ORDER WITH CONSTANT COEFFICIENTS*

BY

J. F. RITT

Introduction

Pincherle, in his classic writings on distributive operations,† has shown that the inversion of distributive operations can be made to depend on the solution of linear differential equations of infinite order. The same result was reached by Bourlet,‡ who furthermore undertook the study of such differential equations, but in spite of the title of his memoir, no results appear which would mark a genuine departure from equations of finite order to those whose orders are infinite. In fact, as far as I am aware, no intensive study of such equations has ever been made.

In the present paper such a study is made for the important case where the coefficients are constants, and are subject to one further condition. The first part is given up to the theory of the " entire differential operator of genus zero,"

$$A = \left(1 - \frac{D}{a_1}\right)\left(1 - \frac{D}{a_2}\right)\cdots\left(1 - \frac{D}{a_n}\right)\cdots,$$

where D denotes differentiation and where the constants a_n are such that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|}$$

is convergent. As far as I know, this operator has never been studied before. Its most notable property is that its domain of applicability consists of all analytic functions. That this property belongs to the linear differential

* Presented to the Society, under a different title, April 29, 1916.

† S. Pincherle, *Operazione distributive*, p. 136; *Mémoire sur le calcul fonctionnel distributif*, Mathematische Annalen, vol. 49 (1897), p. 356; *Equations et opérations fonctionnelles*, Encyclopédie des Sciences Mathématiques, II, 26, p. 25.

‡ C. Bourlet, *Sur les opérations en général et les équations linéaires différentielles d'ordre infini*. Annales de l'école normale, ser. 3, vol. 33 (1897).

development of A was probably known to Bourlet, although he made no explicit mention of the fact.

The second part contains a discussion of the most general solution of the equation $A\phi(z) = 0$. The general properties of the solutions are first obtained—the most striking being, perhaps, that the solutions are all uniform—and the analytical representation of the solutions is discussed. In § 11 an application is made to the theory of analytic prolongation, there being obtained, from an entirely new point of view, a known sufficient condition that the circle of convergence of a power series be a natural boundary.

I hope to present later the results of an investigation which I am now conducting on the inversion of other classes of operators.

In notation, I have followed Pincherle, in the main, using capital Roman letters for operators, and small Greek letters for functional symbols. The sum and the product of two distributive operations are defined by the equations

$$(A + B)\phi(z) = A\phi(z) + B\phi(z), \qquad BA\phi(z) = B[A\phi(z)],$$

respectively. Other questions of notation will be handled as they arise.

Professor Fite has read this paper, as well as the preceding one, and he is responsible for numerous improvements in both. I welcome this opportunity to thank him.

Part 1. The entire differential operator of genus zero

1. The operator as an infinite product. The reader is familiar with the operator

$$A_n = \left(1 - \frac{D}{a_1}\right)\left(1 - \frac{D}{a_2}\right)\cdots\left(1 - \frac{D}{a_n}\right),$$

where $a_1, a_2, \cdots, a_n$, are any real or complex numbers except zero. We shall call each operator $(1 - D/a_n)$ a "factor," and each a_n a "zero" of A_n. The domain of applicability of A_n consists of all functions which have n derivatives. The order of the factors of A_n is immaterial. It is legitimate to develope A_n as a polynomial in D and to apply it as a linear differential operator.

We shall define now the operator

$$A = \left(1 - \frac{D}{a_1}\right)\left(1 - \frac{D}{a_2}\right)\cdots\left(1 - \frac{D}{a_n}\right)\cdots.$$

We are to have

$$A\phi(z) = \lim_{n \doteq \infty} A_n\,\phi(z),$$

so that $A\phi(z)$ will have a meaning provided that $\phi(z)$ has derivatives of all orders, and that the limit involved in the definition exists. Thus, to operate with A will be to operate first with $(1 - D/a_1)$, to apply $(1 - D/a_2)$

to the result, etc. When $A\phi(z)$ has a meaning, we may speak of it as being convergent. What we shall mean by the convergence being uniform in a given domain is obvious. We are interested here in the case where

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|}$$

is convergent. In that case, we shall call A, by reason of an obvious analogy, an "entire differential operator of genus zero." We shall prove, concerning such an operator A, the

THEOREM I. *If $\phi(z)$ is holomorphic in a given domain, $A\phi(z)$ converges in that domain, the convergence being uniform in every closed and bounded domain interior to the given domain.*

It follows from a well-known theorem of Weierstrass that $A\phi(z)$ converges to an analytic function.

We shall need the following statement of Taylor's theorem:

LEMMA I*a*. *Denoting by e^{aD} the operator*

$$1 + aD + \frac{a^2 D^2}{2!} + \cdots + \frac{a^n D^n}{n!} + \cdots,$$

of which the manner of application is evident, and operating with e^{aD} upon the function $\phi(z)$, analytic for $|z| \leqq r$, where $r > |a|$, we obtain $\phi(z + a)$, which is analytic at least for $|z| \leqq r - |a|$.

We must have also the following lemma:

LEMMA I*b*. *If $\phi(z)$, analytic for $|z| \leqq r$, has h as the upper bound of its modulus for $|z| \leqq r$, then, if $|z_1|$ and $|z_2|$ are each less than $r - \delta$, where $0 < \delta < r$, we have*

$$|\phi(z_2) - \phi(z_1)| < \frac{hr|z_2 - z_1|}{\delta^2}.$$

In short, the derivative of $\phi(z)$ will have as a majorant, for $|z| \leqq r$, the function $hr/(r - |z|)^2$, so that, for $|z| < r - \delta$,

$$\left|\frac{d\phi(z)}{dz}\right| < \frac{hr}{\delta^2},$$

and

$$|\phi(z_2) - \phi(z_1)| = \left|\int_{z_1}^{z_2} \frac{d\phi(z)}{dz}\,dz\right| < \frac{hr|z_2 - z_1|}{\delta^2}.$$

We shall consider now the convergence of $A\phi(z)$ in the neighborhood of any given point, which point we shall take as the origin for simplicity. Thus, let

$$\phi(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots,$$

be regular for $|z| \leqq r$. Now, let

$$\bar{A} = \left(1 + \frac{D}{|a_1|}\right)\left(1 + \frac{D}{|a_2|}\right)\cdots\left(1 + \frac{D}{|a_n|}\right)\cdots,$$

and let $\bar{A}_n$ be the operator formed by the first n factors of $\bar{A}$. Also, let

$$\bar{\phi}(z) = |b_0| + |b_1|z + |b_2|z^2 + \cdots + |b_n|z^n + \cdots.$$

The majorant $\bar{\phi}(z)$, like $\phi(z)$, is regular for $|z| \leqq r$. Take any $\delta > 0$, where $3\delta < r$. We shall study first the effect of operating on $\bar{\phi}(z)$ with $\bar{A}$, for $|z| < r - 3\delta$. Take any ϵ, positive, but less than δ, and choose an integer m such that, if $n \geqq m$,

$$\sum_{n+1}^{n+p} \frac{1}{|a_q|} < \epsilon,$$

for every positive integer p. The coefficient of any power of D after the first in the development of

$$\left(1 + \frac{D}{|a_{n+1}|}\right)\left(1 + \frac{D}{|a_{n+2}|}\right)\cdots\left(1 + \frac{D}{|a_{n+p}|}\right)$$

is less than the coefficient of the same power of D in the development of

$$e^{\left(\frac{1}{|a_{n+1}|} + \frac{1}{|a_{n+2}|} + \cdots + \frac{1}{|a_{n+p}|}\right) D}.$$

Hence, for $|z| \leqq r - \delta$ and for $n \geqq m$, we have by Lemma I*a*,

$$\begin{aligned}(\bar{A}_{n+p} - \bar{A}_n)\bar{\phi}(|z|) &= \left[\left(1 + \frac{D}{|a_{n+1}|}\right)\cdots\left(1 + \frac{D}{|a_{n+p}|}\right) - 1\right]\bar{A}_n\bar{\phi}(|z|) \\ &< \bar{A}_n\bar{\phi}(|z| + \epsilon) - \bar{A}_n\bar{\phi}(|z|).\end{aligned}$$

Let n have the value m for a moment. Since $\bar{A}_m\bar{\phi}(|z|)$ is bounded, for $|z| \leqq r$,

$$\bar{A}_m\bar{\phi}(|z| + \epsilon) - \bar{A}_m\bar{\phi}(|z|)$$

is bounded for $|z| \leqq r - \delta$. Then, by the above inequality, all the functions $\bar{A}_{m+p}\bar{\phi}(|z|)$, or, what is the same, all the functions $\bar{A}_n\bar{\phi}(|z|)$, for $n \geqq m$, have a common upper bound h for $|z| \leqq r - \delta$.* Then, by Lemma I*b*, we have for $|z| \leqq r - 3\delta$, for *any n greater than or equal to m*, and for any p,

$$(\bar{A}_{n+p} - \bar{A}_n)\bar{\phi}(|z|) < \frac{h(r-\delta)\epsilon}{\delta^2} < \frac{hr\epsilon}{\delta^2}.$$

Since the coefficient of any power of D in $\bar{A}_n$ is not less than the absolute

* It is essential to bear in mind that although we used an ϵ in determining h, this h depends really only on the sequence of functions $\bar{A}_n\bar{\phi}(|z|)$ and can be used again and again while ϵ is sent to zero by increasing m.

value of the corresponding coefficient in A_n, it is clear that $\bar{A}_n \bar{\phi}(z)$ will be a majorant of $A_n \phi(z)$, so that

$$D^q A_n \bar{\phi}(|z|) \geqq |D^q A_n \phi(z)|, \tag{1}$$

for $z \leqq r$ and for every q. Now we have already seen that

$$(\bar{A}_{n+p} - \bar{A}_n)\bar{\phi}(|z|) = \left[\left(1 + \frac{D}{|a_{n+1}|}\right) \cdots \left(1 + \frac{D}{|a_{n+p}|}\right) - 1\right] \bar{A}_n \bar{\phi}(|z|)$$

and that

$$(A_{n+p} - A_n)\phi(z) = \left[\left(1 - \frac{D}{a_{n+1}}\right) \cdots \left(1 - \frac{D}{a_{n+p}}\right) - 1\right] A_n \phi(z).$$

Comparing the corresponding coefficients in the developments of $\bar{A}_{n+p} - \bar{A}_n$ and $A_{n+p} - A_n$, and taking account of (1), we have

$$|(A_{n+p} - A_n)\phi(z)| \leqq (\bar{A}_{n+p} - \bar{A}_n)\bar{\phi}(|z|) < \frac{hr\epsilon}{\delta^2}, \tag{2}$$

for $|z| \leqq r - 3\delta$ and for $n \geqq m$. Since h and δ are fixed numbers, and since ϵ can be made arbitrarily small by a proper choice of m, it is clear that $A\phi(z)$ converges uniformly for $|z| \leqq r - 3\delta$. It follows immediately that $A\phi(z)$ converges to a holomorphic function in any domain in which $\phi(z)$ is holomorphic. Also it can easily be shown by means of the Heine-Borel theorem that the convergence is uniform in every closed and bounded domain interior to the domain of regularity of $\phi(z)$. Thus Theorem I is proved.*

It would be natural now to state this theorem for any domain on a Riemann surface. To avoid whatever may be vague in the concept of the most general such surface, we limit ourselves to saying, that if $\phi(z)$ is multiform, $A\phi(z)$ converges uniformly on any curve of finite length along which $\phi(z)$ can be prolonged, whether the curve intersects itself or not. This fact will be very useful to us later.

In the case where $\sum 1/|a_n|$ is divergent, it is easily seen that $\bar{A}\bar{\phi}(z)$ diverges to $+\infty$ for every positive value of z less than r. Thus, the condition that $\sum 1/|a_n|$ be convergent plays practically the same rôle in the present theory as it does in that of the infinite product.

* Theorem I can be extended to the case where each differentiation is preceded by a multiplication by an analytic function $\zeta_n(z)$ and is followed by a multiplication by an analytic function $\chi_n(z)$, provided all the functions $\zeta_n(z)$, $\chi_n(z)$ have a common upper bound for their moduli in the given domain. Operators in which each $\zeta_n(z)$ is unity and in which $\chi_n(z)$ does not vary with n, may be reduced to the form of A, above, by a suitable change of variable.

Theorem I indicates, and further developments will emphasize, the analogy between the theory of the operator A, and that of the ordinary infinite product. One distinction will, however, arise. We shall see, in fact, that although an absolutely convergent infinite product cannot vanish unless one of its factors does, $A\phi(z)$ may very well converge to zero without any $A_n\phi(z)$ being identically zero.

Throughout the rest of this paper, A will stand for an entire differential operator of genus zero.

2. **Degree of convergence.*** From the inequality (2) above, we infer, since h, r, and δ are fixed once for all, and since ϵ can be taken as $\sum_{n+1}^{\infty} 1/|a_q|$, the result:

THEOREM II. *In any closed and bounded domain interior to the domain of regularity of $\phi(z)$, the convergence of $A\phi(z)$ is at least as rapid as that of $\sum 1/|a_n|$; that is, the ratio of $|A\phi(z) - A_n\phi(z)|$ to $\sum_{n+1}^{\infty} 1/|a_q|$ is ultimately less than some finite number.*

This is evident for a sufficiently small neighborhood of any point, and the extension to the larger domain is immediate.

An interesting special case presents itself when $A\phi(z)$ converges to zero for all values of z.† Suppose $\phi(z)$ regular for $|z| \leqq r$. Preassigning some positive integer m, take $\delta > 0$ such that $2m\delta < r$, and so choose s that, for $n > s$,

$$\sum_{n+1}^{\infty} \frac{1}{|a_q|} < \delta .$$

Let ϵ_n be the maximum of $|A_n\phi(z)|$ for $|z| = r$. Then a majorant of $A_n\phi(z)$ will be $r\epsilon_n/(r - |z|)$. Hence, by what we have seen in the proof of Theorem I, we must have, for $|z| \leqq r - 2\delta$, and for $n > s$,

$$|A_{n+p}\phi(z) - A_n\phi(z)| \leqq \left[\left(1 + \frac{D}{|a_{n+1}|}\right)\cdots\left(1 + \frac{D}{|a_{n+p}|}\right) - 1\right]\frac{r\epsilon_n}{r - |z|}$$
$$< \frac{r\epsilon_n \sum_{n+1}^{\infty} 1/|a_q|}{\delta^2} .$$

Thus, since $A_{n+p}\phi(z)$ approaches zero as p increases,

$$|A_n\phi(z)| \leqq \frac{r\epsilon_n \sum_{n+1}^{\infty} 1/|a_q|}{\delta^2} ,$$

for $|z| \leqq r - 2\delta$ and for $n > s$. Carrying out this process m times, with a few slight modifications, we find, finally,

$$|A_n\phi(z)| \leqq \frac{r^m \epsilon_n \left(\sum_{n+1}^{\infty} 1/|a_q|\right)^m}{\delta^{2m}} , \tag{3}$$

for $|z| \leqq r - 2m\delta$. Since, when δ is once fixed, $\sum_{n+1}^{\infty} 1/|a_q|$ becomes infinitesimal compared to it as n increases, we infer from (3) the two theorems which follow.

THEOREM III. *If $A\phi(z) = 0$, the modulus of $A_n\phi(z)$, at any point, becomes infinitesimal as n increases, compared to the maximum modulus of*

* This section can be omitted in a first reading.

† See the final remarks in the footnote on p. 31.

$A_n \phi(z)$ on any circle about that point as center, on and within which $\phi(z)$ is regular.

To a certain extent, this fact is not surprising, for it is well known that, in a closed and bounded domain, the modulus of an analytic function assumes its maximum value on the boundary. It is from the intensity of the phenomenon that the theorem derives its interest.

THEOREM IV. *If $A\phi(z) = 0$, $|A_n \phi(z)|$ becomes less in absolute value than any preassigned power of $\sum_{n+1}^{\infty} 1/|a_q|$ as n increases.*

This is seen, from (3), for a neighborhood of every point, and can be extended immediately to any closed and bounded domain of regularity.

3. **The operator A as a linear differential operator.** A can be developed formally into a linear differential operator (A), of infinite order. How (A) is to be applied to an analytic function $\phi(z)$, and what we shall mean by the convergence or uniform convergence of $(A)\phi(z)$ in a given domain, are matters on which it is unnecessary to dwell.

THEOREM V. *If $\phi(z)$ is holomorphic in a given domain, $(A)\phi(z)$ converges to $A\phi(z)$ in that domain, the convergence being absolute, and uniform in every closed and bounded domain interior to the given domain.*

Let $(A)_n$ be the operator formed by the first $n+1$ terms of (A).* Choosing first an $\epsilon > 0$, take m such that for $n > m$,

$$\bar{A}_{n+p}\,\bar{\phi}(|z|) - \bar{A}_n\,\bar{\phi}(|z|) < \tfrac{1}{2}\epsilon, \tag{4}$$

for $|z| \leqq r$, and for every p. It is easy to see that

$$(\bar{A})_{n+q}\,\bar{\phi}(|z|) - \bar{A}_n\,\bar{\phi}(|z|) < \epsilon, \tag{5}$$

for $|z| \leqq r$ and for every q. In short, as p increases, the first $n+q+1$ coefficients in the development of $\bar{A}_{n+p}$ approach the corresponding coefficients in $(\bar{A})_{n+q}$, and $\bar{A}_{n+p}$ will also have terms of higher order, with positive coefficients. Thus, if there is a q for which (5) is not satisfied, (4) will not hold either, for sufficiently large values of p.

Now $(\bar{A})_{n+q} - \bar{A}_n$ is the linear differential operator formed by the first $n+q+1$ terms in the development of

$$\left(1+\frac{D}{|a_1|}\right)\left(1+\frac{D}{|a_2|}\right)\cdots\left(1+\frac{D}{|a_n|}\right)\left\{\left[\left(1+\frac{D}{|a_{n+1}|}\right)\cdots\left(1+\frac{D}{|a_{n+q}|}\right)\cdots\right]-1\right\},$$

so that the coefficients in the development of $(\bar{A}_{n+q}) - \bar{A}_n$ are positive, and are each not less than the absolute values of the corresponding coefficients in $(A)_{n+q} - A_n$. Hence, referring to (5), we have

$$|(A)_{n+q}\,\phi(z) - A_n\,\phi(z)| \leqq \epsilon,$$

* Observe that $(A)_n$ is not the development of A_n.

for $|z| \leqq r$, for $n > m$ and for every q. From this last inequality, the truth of Theorem V for $|z| \leqq r$ follows without difficulty. The extension to the larger domain is immediate. The absolute convergence follows from the convergence of $(\bar{A})\bar{\phi}(z)$.

COROLLARY I. *The order of the factors of A is immaterial.*

In short, whatever be the order of the factors, the same (A) will result.

COROLLARY II. *A is commutative with any power of D.*

This is clearly a property of (A), and hence one of A.*

That $(A)\phi(z)$ always converges was very probably known to Bourlet, although he failed to state the fact explicitly.† Bourlet would have appealed, for the proof, to a theorem by Poincaré on the coefficients of an entire function. On the other hand, our method of proof has put us in a position to prove

POINCARE'S THEOREM.‡ *If*

$$\phi(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots$$

is an entire function of genus zero, and if a is any number whatsoever, then

$$\lim_{n=\infty} n!\, a^n c_n = 0.$$

Let us operate on $1/(1 - az)$ with

$$c_0(A) = c_0 + c_1 D + c_2 D^2 + \cdots + c_n D^n + \cdots$$

at the point $z = 0$. We get, since

$$D^n\left(\frac{1}{1 - az}\right)_{z=0} = n!\, a^n ,$$

$$c_0(A)\left(\frac{1}{1 - az}\right)_{z=0} = c_0 + ac_1 + 2!\, a^2 c_2 + \cdots + n!\, a^n c_n + \cdots .$$

Poincaré's theorem follows from the convergence of the series above. This theorem is only a special case of a theorem which Poincaré proved, by an entirely different method, for entire functions of any finite genus.§

The two theorems which follow will be of frequent use.

THEOREM VI. *Given a domain d, a closed and bounded domain d_1 interior to d, and a positive number ϵ, we can find a positive number h such that $|A\phi(z)| < \epsilon$ in d_1 when $\phi(z)$ is holomorphic, and less in absolute value than h, throughout d.*

If $\phi(z)$ is regular for $|z| \leqq r$, $D^n \phi(z)$ will have as a majorant

$$n!\, hr/(r - |z|)^{n+1},$$

* Cf. Pincherle, *Operazione Distributive*, p. 119.

† Bourlet, loc. cit., pp. 159 and 161.

‡ In the proof, we assume that $c_0 \neq 0$, but this is not essential.

§ See Borel, *Fonctions Entières*, Chapter III.

so that, if $0 < \delta < r$,

$$|D^n \phi(z)| \leqq \frac{n!\, rh}{\delta^{n+1}}$$

for $|z| \leqq r - \delta$. Then, using the expression for (A) above,

$$|(A)\phi(z)| \leqq \frac{rh}{\delta}\left(1 + \frac{|c_1|}{\delta} + \frac{2!|c_2|}{\delta^2} + \cdots + \frac{n!|c_n|}{\delta^n} + \cdots\right).$$

Referring to the proof of Poincaré's theorem, we see that the series within the parentheses is convergent, so that $|(A)\phi(z)|$ goes to zero with h, for $|z| \leqq r - 2\delta$. This proves the theorem for a neighborhood of any point, and the extension to the domain d_1 is immediate.

THEOREM VII. *Given the domains d and d_1 above, and any two positive numbers, ϵ and h, we can determine a positive number η such that, if $\sum 1/|a_n| < \eta$, we have*

$$|A\phi(z) - \phi(z)| < \epsilon$$

in d_1, provided $\phi(z)$ is holomorphic, and less in absolute value than h, throughout d.

The proof, which is very simple, we indicate briefly. If $\phi(z)$ is regular for $z \leqq r$, it will have, in the neighborhood of the origin, a majorant $\overline{\phi}(z)$, which is less in absolute value than h for $|z| \leqq r$. Then, if $\eta < \delta < r$,

$$|A\phi(z) - \phi(z)| \leqq |\overline{A}\overline{\phi}(|z|) - \overline{\phi}(|z|)| < (e^{\eta D} - 1)\overline{\phi}(|z|) < \frac{hr\eta}{\delta^2}$$

for $z \leqq r - \delta$. This inequality leads readily to the theorem.

We shall need later, in considering multiform functions, the following modification of Theorem VII.

THEOREM VII$_1$. *Given a function $\phi(z)$, a curve of finite length on which, inclusive of the extremities, $\phi(z)$ is analytic, and any $\epsilon > 0$, we can find an $\eta > 0$, such that, when $\sum 1/|a_n| < \eta$, we have*

$$|A\phi(z) - \phi(z)| < \epsilon$$

along the curve in question.

4. **Distributivity of the operator** A. From the fact that every A_n is distributive, it follows that A is also distributive; that is, as long as only a finite number of functions are involved. To extend the distributivity of A to the sum of an infinite number of functions, we prove the theorem which follows:

THEOREM VIII. *If*

$$\psi(z) = \sum_{n=1}^{\infty} \phi_n(z)$$

is uniformly convergent in a given area, each $\phi_n(z)$ being holomorphic in that area, we have, in the given area,

$$A\psi(z) = \sum_{n=1}^{\infty} A\phi_n(z),$$

and the series in the second member of the last equation is uniformly convergent in every closed and bounded domain interior to the given area.

The proof involves the application of a principle stated by Pincherle for all distributive operations.* We have

$$A\psi(z) = A\sum_{1}^{n}\phi_q(z) + A\sum_{n+1}^{\infty}\phi_q(z) = \sum_{1}^{n}A\phi_q(z) + A\sum_{n+1}^{\infty}\phi_q(z).$$

But from the uniform convergence of $\sum \phi_n(z)$ and from Theorem VI, we see that

$$A\sum_{n+1}^{\infty}\phi_q(z)$$

goes to zero uniformly in any closed and bounded domain interior to the given area as n increases indefinitely. The theorem is proved.

5. **Multiplication and factorization of operators.**

THEOREM IX. *If A and B are two entire differential operators of genus zero, their product BA† will also be an entire operator of genus zero, and its factors will be the combined factors of A and of B.*

Obviously, the theorem will be proved if we can show that

$$BA\phi(z) = \lim_{n=\infty} B_n A_n \phi(z)$$

for every analytic $\phi(z)$. We have, identically,

$$BA\phi(z) - B_n A_n \phi(z) = B(A - A_n)\phi(z) + (B - B_n)A_n\phi(z).$$

Now, in any closed and bounded domain, $(A - A_n)\phi(z)$ goes to zero uniformly as n increases, so that, by Theorem VI, $B(A - A_n)\phi(z)$ approaches zero also. We have

$$(B - B_n)A_n\phi(z) = \left\{\left[\left(1 - \frac{D}{b_{n+1}}\right)\left(1 - \frac{D}{b_{n+2}}\right)\cdots\right] - 1\right\}B_n A_n \phi(z),$$

where the significance of the numbers b is evident. Now, since $B_n A_n \phi(z)$, as we know beforehand, approaches a limit uniformly as n increases, it must stay bounded, as n increases, in any small domain of regularity of $\phi(z)$. Thus, the conditions of Theorem VII hold, and $(B - B_n)A_n\phi(z)$ is seen to approach zero. The theorem is proved.

COROLLARY I. *The product of any finite number of operators of genus zero can be formed by collecting the factors of the separate operators.*

COROLLARY II. *The operators A and B, above, are commutative.*

COROLLARY III. *It is legitimate to group the factors of A in any manner, writing A as the product of a finite or an infinite number of operators, each containing a finite or an infinite number of the factors of A.*

* Mathematische Annalen, vol. 49 (1897), p. 349.

† See the remarks on notation on p. 28. Note that we do not refer to the *formal* product of A and B.

For the case of separation into a finite number of operators, this follows directly from Corollary I. In the case of an infinite number of operators, a few other simple considerations are necessary.

Part 2. The homogeneous equation of genus zero

6. **Comparison with the equation of finite order.** The object of the second part of this paper is to discuss the most general analytic function $\phi(z)$ such that

$$A\phi(z) = 0. \tag{1}$$

We shall call (1) the "homogeneous equation of genus zero."

If $\phi(z)$ satisfies (1) in an arbitrarily small neighborhood, it will satisfy (1) in its entire domain of existence, for, by Theorem I, $A\phi(z)$ is analytic on every curve along which $\phi(z)$ can be prolonged.

Evidently, the solutions of every equation

$$A_n \phi(z) = 0 \tag{2}$$

are solutions of (1). Also, if the zeros of A are

$$a_1, a_2, \cdots, a_n, \cdots,$$

of multiplicities

$$p_1, p_2, \cdots, p_n, \cdots$$

respectively,* we have the

Theorem X. *If the series*

$$\phi(z) = \sum_{n=1}^{\infty} e^{a_n z}(c_{n,0} + c_{n,1} z + \cdots + c_{n,p_n-1} z^{p_n-1})$$

is uniformly convergent in some area, it satisfies (1) *in that area.*

This follows from Theorem VIII, since each of the terms of the series is a solution of (1).

Theorem X indicates that (1) has solutions which satisfy no (2), and that the general solution of (1) contains an infinite number of arbitrary constants. It does not furnish a rigorous proof of either of these facts, for we cannot say, as yet, that all of the parameters in the series above are essential. However, we shall show that (1) has solutions which satisfy no (2),† and indeed, in a way which will reveal a striking difference between the two kinds of equations. The solutions of (2) are all entire functions. We shall show that the solutions of (1) may have singularities in the finite part of the plane. The series

$$e^z + e^{2^2 z} + \cdots + e^{n^2 z} + \cdots,$$

* Note that

$$\sum_{n=1}^{\infty} \frac{p_n}{|a_n|}$$

is convergent.

† Cf. the second paragraph in the footnote on p. 31.

which is uniformly convergent in any domain in which the real part of z is less than some negative number, is, in such a domain, a solution of (1) when the zeros of A are $1^2, 2^2, \cdots, n^2, \cdots$. As z increases towards zero on the axis of reals, the terms of the series will each approach unity, and the series will tend towards $+\infty$. Thus, the function defined by the series cannot be regular for $z = 0$.

Still, the theorem which follows establishes a close connection between (1) and (2).

THEOREM XI. *The solutions of* $A\phi(z) = 0$ *and the successive integrals of such solutions, are all uniform functions.*

It is a question of showing that every analytic $\phi(z)$ such that

$$AD^p \phi(z) = 0,$$

p being any integer, is uniform. Suppose then that the equation above has a multiform solution $\phi(z)$, and let c be a point at which $\phi(z)$ has more than one value. By the second corollary of Theorem V, we have also

$$D^p A\phi(z) = 0.$$

There must exist a curve, beginning and ending at c, along which $\phi(z)$ can be prolonged, and which leads from one of the values at c to a second.

In operating upon $\phi(z)$ with A, it is permissible, by the third corollary of Theorem IX, to apply first the operator

$$\left(1 - \frac{D}{a_{n+1}}\right)\left(1 - \frac{D}{a_{n+2}}\right)\cdots,$$

which we shall denote by $A_n^{-1} A$, and to follow with A_n. Thus, $A_n^{-1} A\phi(z)$, since it vanishes when operated upon with $D^p A_n$, is uniform along the curve described above. As n increases indefinitely, we see, by Theorem VII_1, that $A_n^{-1} A\phi(z)$ approaches $\phi(z)$ along that curve, so that $\phi(z)$ cannot have two distinct values at c. The theorem is proved.

THEOREM XII. *If a solution of* $A\phi(z) = 0$ *is analytic on the entire circumference of a circle, it is analytic throughout the interior of the circle; in particular, no solution of* $A\phi(z) = 0$ *can have an isolated singularity.**

It will suffice, for the proof, to show that $\phi(z)$ cannot have a Laurent development, at any point, in which negative powers are actually present. Suppose, then, that

$$\phi(z) = b_0 + b_1(z - z_1) + b_2(z - z_1)^2 + \cdots + b_n(z - z_1)^n + \cdots$$
$$+ b_{-r}(z - z_1)^{-r} + \cdots + b_{-n}(z - z_1)^{-n} + \cdots,$$

where $b_{-r} \neq 0$, the development being valid in some ring about z_1. Now, since

$$(A)\phi(z) = \phi(z) + c_1 D\phi(z) + \cdots + c_n D^n \phi(z) + \cdots$$

* We except the case of an isolated singularity at infinity.

is uniformly convergent in the ring, it is possible, by Weierstrass's theorem, to get the development of $(A)\phi(z)$ at z_1 by adding up the developments of its separate terms. It is easily seen that the development of $(A)\phi(z)$ must contain the term $b_{-r}(z-z_1)^{-r}$, so that $(A)\phi(z)$ cannot be identically zero. This proves the theorem.

This theorem is also a direct consequence of Theorem XI; for if the Laurent series above contained negative powers, a sufficient number of integrations would introduce a logarithmic term, and one of the integrals of $\phi(z)$ would be multiform.

7. **Determination and identification of the formal development.** Judging by Theorem X, one might suspect that every solution of $A\phi(z)=0$ is expressible, in all or in part of its domain of existence,* by a uniformly convergent development

$$\phi(z)=\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}e^{a_nz}(c_{n,0}+c_{n,1}z+\cdots+c_{n,p_n-1}z^{p_n-1}). \tag{3}$$

We shall show how this development can be determined when it exists.

Denote by $(1-D/a_n)^{-r}A$, where $r\leqq p_n$, the operator

$$\left(1-\frac{D}{a_1}\right)^{p_1}\left(1-\frac{D}{a_2}\right)^{p_2}\cdots\left(1-\frac{D}{a_n}\right)^{p_n-r}\left(1-\frac{D}{a_{n+1}}\right)^{p_{n+1}}\cdots.$$

Let $\zeta(z)$ be the entire function which results on substituting z for D in the expression for A, and let $\zeta^{(n)}(z)$ be the nth derivative of $\zeta(z)$.

Let us operate on both sides of (3) with $(1-D/a_n)^{-1}A$. Clearly, the only contribution from the second member of (3) will be from the nth term, and then only from $c_{n,p_n-1}z^{p_n-1}e^{a_nz}$. We find, by direct calculation,

$$\left(1-\frac{D}{a_n}\right)^{p_n-1}z^{p_n-1}e^{a_nz}=(-1)^{p_n-1}\frac{(p_n-1)!}{a_n^{p_n-1}}e^{a_nz}$$

and

$$\left(1-\frac{D}{a_n}\right)^{-p_n}Ae^{a_nz}=e^{a_nz}\left(1-\frac{a_n}{a_1}\right)^{p_1}\cdots\left(1-\frac{a_n}{a_{n-1}}\right)^{p_{n-1}}\left(1-\frac{a_n}{a_{n+1}}\right)^{p_{n+1}}\cdots.$$

The second member of the last equation is readily expressed by means of the p_nth derivative of $\zeta(z)$, calculated according to Leibnitz's theorem. We find thus,

$$\left(1-\frac{D}{a_n}\right)^{-p_n}Ae^{a_nz}=\frac{(-1)^{p_n}a_n^{p_n}}{p_n!}\zeta^{(p_n)}(a_n)e^{a_nz}.$$

Hence

$$\left(1-\frac{D}{a_n}\right)^{-1}A\phi(z)=-\frac{a_n\zeta^{(p_n)}(a_n)}{p_n}c_{n,p_n-1}e^{a_nz},$$

* That is, in a two-dimensional part.

or

$$c_{n,\,p_n-1} = -\frac{p_n\, e^{-a_n z}}{a_n\, \zeta^{(p_n)}(a_n)}\left(1-\frac{D}{a_n}\right)^{-1} A\phi(z).$$

We find, in a similar manner, for the other coefficients in the nth term, the recursion formula

$$c_{n,\,p_n-r} = (-1)^r \frac{p_n!\, e^{-a_n z}}{(p_n - r)!\, a_n^r\, \zeta^{(p_n)}(a_n)}\left(1-\frac{D}{a_n}\right)^{-r} A\,[\phi(z) - e^{a_n z}\,(c_{n,\,p_n-r+1}\, z^{p_n-r+1} + \cdots + c_{n,\,p_n-1}\, z^{p_n-1})]. \tag{4}$$

Formula (4) shows that if a uniformly convergent development (3) exists, it will be unique. The question arises as to whether (3) represents $\phi(z)$ if it is uniformly convergent in a part of the domain of existence of $\phi(z)$. The reply is affirmative. We shall prove, indeed, with greater generality, the following theorem:

THEOREM XIII. *If there exists a number h such that*

$$\left|\sum_1^n u_q\right| < h$$

for every n, in an area contained in the domain of existence of $\phi(z)$, the development (3) converges uniformly to $\phi(z)$ in every closed and bounded domain d_1 interior to that area.

One of the points involved in the proof will be of great importance later, and is of interest in itself. We therefore stop to give it as a separate theorem.

THEOREM XIV. *The result obtained by operating on a solution of $A\phi(z) = 0$ with all but a finite number of the factors of A, is identical with the result obtained by operating formally on the development (3) of that solution, whether the development is valid or not.*

We are to operate on a solution, $\phi(z)$, and on its development (3), with

$$\left(1-\frac{D}{a_1}\right)^{r_1}\cdots\left(1-\frac{D}{a_m}\right)^{r_m}\left(1-\frac{D}{a_{m+1}}\right)^{p_{m+1}}\cdots\left(1-\frac{D}{a_n}\right)^{p_n}\cdots,$$

where $r_i \leqq p_i$, $i = 1, 2, \cdots, m$. The contribution from the development (3) will come only from the first m terms, $\sum_1^m u_q$. It suffices then to show that we get identical results when we operate on the solution and on its development with

$$\left(1-\frac{D}{a_{m+1}}\right)^{p_{m+1}}\cdots\left(1-\frac{D}{a_n}\right)^{p_n}\cdots;$$

the application of the finite number of remaining factors will not disturb the equality.

The finite series $\sum_1^m u_q$ is a solution of $A\phi(z) = 0$, and thus admits of development into a series (3), by the method exposed above. Since we know beforehand, from the very form of the solution, that a valid development

exists, and since we have seen that such a development is unique, the process above will conduct us again to the finite series $\sum_1^m u_q$. The function $\phi(z) - \sum_1^m u_q$ is also a solution of $A\phi(z) = 0$, and has a development (3). Since the operations effected in obtaining the development (3) are distributive, and since the first m terms in the developments of $\phi(z)$ and of $\sum_1^m u_q$ coincide, the first m terms in the development of $\phi(z) - \sum_1^m u_q$ must be identically zero. Thus, drawing from (4) the coefficient of $e^{a_i z}$ in that development, we get, for $i \leqq m$,

$$\left(1 - \frac{D}{a_i}\right)^{-p_i} A\left[\phi(z) - \sum_1^m u_q\right] = 0.$$

Let

$$\xi_m(z) = \left[\left(1 - \frac{D}{a_{m+1}}\right)^{p_{m+1}} \cdots \left(1 - \frac{D}{a_n}\right)^{p_n} \cdots\right]\left[\phi(z) - \sum_1^m u_q\right].$$

Then

$$\left(1 - \frac{D}{a_2}\right)^{p_2}\left(1 - \frac{D}{a_3}\right)^{p_3} \cdots \left(1 - \frac{D}{a_m}\right)^{p_m} \xi_m(z) = 0,$$

$$\left(1 - \frac{D}{a_1}\right)^{p_1}\left(1 - \frac{D}{a_3}\right)^{p_3} \cdots \left(1 - \frac{D}{a_m}\right)^{p_m} \xi_m(z) = 0, \tag{5}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(1 - \frac{D}{a_1}\right)^{p_1}\left(1 - \frac{D}{a_2}\right)^{p_2} \cdots \left(1 - \frac{D}{a_{m-1}}\right)^{p_{m-1}} \xi_m(z) = 0.$$

It is a simple matter to show that the differential equations (5) have no common integral except

$$\xi_m(z) = 0,$$

and this establishes our theorem, in virtue of the opening paragraph of the proof.

Now we return to Theorem XIII. By hypothesis, $|\phi(z) - \sum_1^m u_q|$ stays bounded, as m increases, in a domain to which d_1 is interior, and, since

$$\xi_m(z) = 0,$$

we see by Theorem VII that $\phi(z) - \sum_1^m u_q$ approaches zero uniformly in d_1. Theorem XIII is proved.*

8. **Second form of the development. Convergence discussion.** To discuss the convergence of the development (3), we shall exhibit $\phi(z)$ in a somewhat different form. Without concerning ourselves with questions of convergence, let us suppose that we have, formally,

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty}\left[\frac{f_{n,1}}{1 - \frac{z}{a_n}} + \frac{f_{n,2}}{\left(1 - \frac{z}{a_n}\right)^2} + \cdots + \frac{f_{n,p_n}}{\left(1 - \frac{z}{a_n}\right)^{p_n}}\right],$$

* The methods of the two theorems just proved permit us easily to show that all solutions common to $A\phi(z) = 0$ and $B\phi(z) = 0$ are solutions of $C\phi(z) = 0$, where the factors of C are the factors common to A and to B.

or

$$1 = \sum_{n=1}^{\infty} \left[\frac{f_{n,1}\zeta(z)}{1 - \dfrac{z}{a_n}} + \frac{f_{n,2}\zeta(z)}{\left(1 - \dfrac{z}{a_n}\right)^2} + \cdots + \frac{f_{n,p_n}\zeta(z)}{\left(1 - \dfrac{z}{a_n}\right)^{p_n}} \right].$$

This being considered as an identity in z, the subsistence of a similar identity in D would lead to a development of $\phi(z)$ in the form

$$\begin{aligned}\phi(z) = \sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \Bigg[f_{n,1}\left(1 - \frac{D}{a_n}\right)^{-1} A\phi(z) + \cdots \\ + f_{n,p_n}\left(1 - \frac{D}{a_n}\right)^{-p_n} A\phi(z) \Bigg].\end{aligned} \tag{6}$$

We say, in fact, that the nth term v_n of (6), taken as a whole, is equivalent to the nth term u_n of (3). If a_n is not a multiple zero of $\zeta(z)$, the proof is almost immediate, and doubtless the general case could be handled by mechanical transformations, but the method which follows, though somewhat indirect, will be less painful.

First we observe, referring to Theorem XIV, that

$$\left(1 - \frac{D}{a_n}\right)^{-p_n} A[\phi(z) - u_n] = 0. \tag{7}$$

We shall prove also that

$$\left(1 - \frac{D}{a_n}\right)^{-p_n} A[\phi(z) - v_n] = 0. \tag{8}$$

This will follow as soon as we have shown that

$$v_n = f_{n,1}\left(1 - \frac{D}{a_n}\right)^{-1} Av_n + \cdots + f_{n,p_n}\left(1 - \frac{D}{a_n}\right)^{-p_n} Av_n. \tag{9}$$

We know that (6) becomes a true identity if we employ the operator A_m instead of A, and the partial fraction development of the reciprocal of the first m factors of $\zeta(z)$ instead of that of $1/\zeta(z)$.* Since

$$\left(1 - \frac{D}{a_n}\right)^{p_n} v_n = 0,$$

we get, for sufficiently large values of m, an identity for v_n consisting of p_n terms, as in (9). As m increases indefinitely, we approach the expression for v_n in (9). Then

$$\begin{aligned} f_{n,1}\left(1 - \frac{D}{a_n}\right)^{-1} A[\phi(z) - v_n] + \cdots \\ + f_{n,p_n}\left(1 - \frac{D}{a_n}\right)^{-p_n} A[\phi(z) - v_n] = 0.\end{aligned} \tag{10}$$

From (10) follows the truth of (8), for if the last term of (10) were not identi-

* We understand in this that equal zeros are written separately.

cally zero, it would be the product of $e^{a_n z}$ by a polynomial in z, while the terms which precede the last would be products of $e^{a_n z}$ by polynomials of lower degree, so that (10) would be impossible.*

From (7) and (8),

$$\left(1 - \frac{D}{a_n}\right)^{-p_n} A(u_n - v_n) = 0.$$

But, also,

$$\left(1 - \frac{D}{a_n}\right)^{p_n} (u_n - v_n) = 0.$$

Let

$$w = \left[\left(1 - \frac{D}{a_{n+r}}\right)^{p_{n+r}} \left(1 - \frac{D}{a_{n+r+1}}\right)^{p_{n+r+1}} \cdots\right] (u_n - v_n).$$

Then

$$\left(1 - \frac{D}{a_1}\right)^{p_1} \cdots \left(1 - \frac{D}{a_{n-1}}\right)^{p_{n-1}} \left(1 - \frac{D}{a_{n+1}}\right)^{p_{n+1}} \cdots \left(1 - \frac{D}{a_{n+r-1}}\right)^{p_{n+r-1}} w = 0$$

and

$$\left(1 - \frac{D}{a_n}\right)^{p_n} w = 0.$$

As in § 7, the last two equations can have no common integral except $w = 0$, and, as r increases, w approaches $u_n - v_n$, so that finally,

$$u_n - v_n = 0,$$

as was to be proved.

The equivalence of (3) and (6) enables us to state sufficient conditions for the convergence of (3) in the entire domain of existence of $\phi(z)$. We say, in fact, that *if*

$$\sum_{n=1}^{\infty} (|f_{n,1}| + |f_{n,2}| + \cdots + |f_{n,p_n}|)$$

is convergent, the development (3) *converges absolutely and uniformly to* $\phi(z)$ *in every closed and bounded domain interior to the domain of existence of* $\phi(z)$.

For the proof, it evidently suffices to show that being given such a closed and bounded domain, there exists a positive number h such that

$$\left|\left(1 - \frac{D}{a_n}\right)^{-r} A\phi(z)\right| < h$$

in that domain, for every n and for $r \leqq p_n$. The existence of such an h can be shown by applying Theorem VII, remembering that $A_m \phi(z)$ approaches zero as m increases indefinitely. Indeed, it is seen that $(1 - D/a_n)^{-r} A\phi(z)$ approaches zero as n increases.

Thus, in virtue of the result of the preceding paper, we can say:

If $\zeta(z)$ has only a finite number of multiple zeros, and if there exists an integer

* Observe that we cannot have $f_{n, p_n} = 0$.

r such that, for $n > r$,

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 + \frac{k}{n},$$

where $k > 2$, *the development* (3) *converges absolutely and uniformly to* $\phi(z)$ *in every closed and bounded domain interior to the domain of existence of* $\phi(z)$.

It is possible, however, to obtain a much weaker condition for the validity of (3). Any solution of $A\phi(z) = 0$ is also a solution of $B\phi(z) = 0$, provided the zeros of A are included among the zeros of B. Let ω be a primitive lth root of unity. Then let B possess, together with every zero a_n of A, the zeros

$$\omega a_n, \ \omega^2 a_n, \ \cdots, \ \omega^{l-1} a_n.$$

If A has some $\omega^s a_n$ as a zero, as well as a_n, the above set of zeros will be repeated in B. Evidently, B as thus determined will be an entire operator of genus zero. We may write it

$$B = \left(1 - \frac{D^l}{a_1^l}\right)^{p_1}\left(1 - \frac{D^l}{a_2^l}\right)^{p_2}\cdots\left(1 - \frac{D^l}{a_n^l}\right)^{p_n}\cdots.$$

Let $\eta(z)$ be the entire function which results on substituting z for D^l in the expression for B. Then, corresponding to the formal development of $1/\eta(z)$ into partial fractions,* we can get a formal development of $\phi(z)$ similar to (6), the operator D^l taking the place of D. Without modifying greatly the discussion in connection with (6), we can show that the new development of $\phi(z)$ is equivalent to (3), provided we collect into one term, those terms of (3) which arise from distinct zeros of A whose lth powers are equal. We may thus state the theorem:

THEOREM XV. *If the sum of the coefficients in the formal development of* $1/\eta(z)$ *is absolutely convergent, the development* (3) *converges absolutely and uniformly to* $\phi(z)$ *in every closed and bounded domain interior to the domain of existence of* $\phi(z)$, *provided we unite those terms of* (3) *which arise from distinct zeros of* A *whose* l*th powers are equal.*†

Now, if

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 + \frac{k}{n},$$

where $k > 0$, we have

$$\left|\frac{a_{n+1}^l}{a_n^l}\right| > 1 + \frac{lk}{n}.$$

Since we can so take l that $lk > 2$, we have the theorem:

THEOREM XVI. *If A has only a finite number of multiple zeros, and if there*

* Observe that two distinct zeros of A may lead to equal zeros of $\eta(z)$.

† In proving that the development yields $\phi(z)$ it is necessary to extend Theorem XIII to the case where the terms of Σu_n are grouped arbitrarily. This extension, like Theorem XIII itself, follows directly from Theorem XIV.

exists an integer r such that, for $n > r$,

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 + \frac{k}{n},$$

where $k > 0$, the development (3) *converges absolutely and uniformly to $\phi(z)$ in every closed and bounded domain interior to the domain of existence of $\phi(z)$.**

9. **Example of a divergent development. Method of summation.** If the moduli of the zeros of A do not increase with sufficient rapidity, the development (3) of a solution of $A\phi(z) = 0$ may diverge everywhere, or may converge in only a portion of the domain of existence of the solution. It will suffice to give an example of the latter case.

Consider the series

$$e^{(1^2-\delta_1)z} - e^{1^2 z} + e^{(2^2-\delta_2)z} - e^{2^2 z} + \cdots + e^{(n^2-\delta_n)z} - e^{n^2 z} + \cdots,$$

where each δ_n is some positive number, less than unity, which will be fixed later. This series is uniformly convergent in any domain in which the real part of z is less than some negative number. In short, $|e^{(n^2-\delta_n)z}|$ and $|e^{n^2 z}|$ will each be less than r^{n^2-1}, where $0 < r < 1$. Also, the series is divergent when the real part of z is positive, for then $|e^{n^2 z}| > 1$. The analytic function which is represented, perhaps in part, perhaps in all of its domain of existence, by the series above, is a solution of $A\phi(z) = 0$ when A has each number n^2

* If the condition of this theorem is not satisfied, it may be possible to interpolate new zeros between those of A in such a way that the condition is satisfied by the new operator. In that case also, every solution of $A\varphi(z) = 0$ has a valid development.

An interesting consequence of this theorem is that *every* equation $A\varphi(z) = 0$ has solutions which are not entire functions. The real and imaginary parts of a_n cannot both stay bounded as n increases. To fix our ideas, suppose that the real parts do not stay bounded, and furthermore suppose that we can select a sequence in which the real parts are positive and increase without limit. Of this sequence we can again select a sequence

$$a_{i_1}, a_{i_2}, \cdots a_{i_n}, \cdots,$$

in which the imaginary parts are all of one sign, say non-negative, in which the real part of a_{i_n} is greater than n, and in which the moduli of the zeros increase so rapidly that any solution of $A\varphi(z) = 0$ built on these zeros has a development (3) which is valid in the entire domain of existence of the solution. Then the series

$$\sum_{n=1}^{\infty} e^{a_{i_n} z}$$

is uniformly convergent in any area in which the real part of z is less than some negative number and in which the imaginary part of z is not negative, and is divergent for $z = 0$. The solution defined by this series has $z = 0$ as a singular point.

In closing this article, it is deserving of notice that the development (6) can be applied to functions which are not solutions of $A\phi(z) = 0$. In fact, it is not difficult to show that if the partial fraction development of $1/\zeta(z)$ converges absolutely to $1/\zeta(z)$, the development (6) of any analytic $\phi(z)$ converges absolutely and uniformly to $\phi(z)$ in every closed and bounded domain in which $\phi(z)$ is regular. Of course, the development obtained will not generally be a series of exponentials.

and $n^2 - \delta_n$ for a zero. Now, our series can be written

$$\sum_{n=1}^{\infty} [e^{(n^2-\delta_n)z} - e^{n^2 z}].$$

By taking each δ_n sufficiently small, we can make $|e^{(n^2-\delta_n)z} - e^{n^2 z}|$ small at pleasure in any bounded domain; for instance, less than $1/n^2$ for $|z| < n$. We find, in this manner, an entire function which is represented only in a limited domain by its formal development (3).

Still, the development (3) is not devoid of significance, even when divergent. It can be converted into a series which represents $\phi(z)$ in its entire domain of existence.

Let us dispense with the exponents p_n and write equal zeros of A separately. Since, by Theorem VII, $A_n^{-1} A\phi(z)$ approaches $\phi(z)$ as n increases, it is clear that we have

$$\begin{aligned}\phi(z) = A_1^{-1} A\phi(z) + (A_2^{-1} A - A_1^{-1} A)\phi(z) + \cdots \\ + (A_n^{-1} A - A_{n-1}^{-1} A)\phi(z) + \cdots,\end{aligned} \tag{11}$$

the convergence being uniform in every closed and bounded domain interior to the domain of existence of $\phi(z)$.

The sum of the first n terms of (11) is $A_n^{-1} A\phi(z)$, and must vanish if we operate on it with A_n. Suppose, for brevity, that A has no multiple zeros. The general case will be easy to handle. Then $A_n^{-1} A\phi(z)$ can be written in the form

$$g_{n,1} e^{a_1 z} + g_{n,2} e^{a_2 z} + \cdots + g_{n,n} e^{a_n z}.$$

Proceeding as in § 7, we find

$$g_{n,r} = \frac{e^{-a_r z}\left(1 - \frac{D}{a_r}\right)^{-1} A\phi(z)}{\left(1 - \frac{a_r}{a_1}\right)\left(1 - \frac{a_r}{a_2}\right)\cdots\left(1 - \frac{a_r}{a_{r-1}}\right)\left(1 - \frac{a_r}{a_{r+1}}\right)\cdots\left(1 - \frac{a_r}{a_n}\right)}.$$

As n increases, $g_{n,r}$ approaches the coefficient of $e^{a_r z}$ in (3). Thus, (11) may be regarded as a summation of (3), when the latter does not serve to represent $\phi(z)$. To justify this point of view completely, we observe that (11) can be obtained directly from (3). In short, Theorem XIV shows that we can obtain $A_n^{-1} A\phi(z)$ by operating formally with $A_n^{-1} A$ on (3).

THEOREM XVII. *The development* (11) *converges absolutely at every point at which* $\phi(z)$ *is regular, and at least with the same rapidity as* $\sum 1/|a_n|$.

This follows easily from the identity

$$(A_n^{-1} A - A_{n-1}^{-1} A)\phi(z) = \frac{1}{a_n} A_n^{-1} A \frac{d\phi(z)}{dz}.$$

10. **Origin of a development (3).** We have seen that a solution of

$A\phi(z) = 0$ can have only one development (3). The important question arises as to whether two different solutions, with non-overlapping domains of existence, may not lead to the same development (3). While we shall settle this question under restricted conditions, we have thus far been unable either to show that different solutions always lead to different developments, or to produce two solutions with the same development. Let us prove the following:

THEOREM XVIII. *If A is such that the development* (3) *of every solution $\phi(z)$ of $A\phi(z) = 0$ converges absolutely in the entire domain of existence of $\phi(z)$, for instance, if the condition of Theorem XV is satisfied, a series*

$$\sum_{n=1}^{\infty} c_n e^{a_n z}$$

can represent only one solution of $A\phi(z) = 0$.

If z_1 and z_3 are any two points, a point z_2 on the straight segment joining them is given by

$$z_2 = z_1 + k(z_3 - z_1) = (1 - k)z_1 + kz_3,$$

where $0 < k < 1$. Then

$$|e^{a_n z_2}| = (|e^{a_n z_1}|)^{1-k}(|e^{a_n z_3}|)^k.$$

It is clear, thus, that $|e^{a_n z_2}|$ lies between $|e^{a_n z_1}|$ and $|e^{a_n z_3}|$. Then, certainly,

$$|e^{a_n z_2}| < |e^{a_n z_1}| + |e^{a_n z_3}|.$$

One sees immediately that if $\sum c_n e^{a_n z}$ converges absolutely at z_1 and z_3, it will converge absolutely and uniformly on the straight segment joining z_1 and z_3. It follows also that if the series converges absolutely in two regions of the plane, it converges uniformly in a parallelogram connecting these two regions. Thus, we can pass by analytic prolongation from one region to the other. Theorem XVIII is proved.

For the general question as to whether a development (3) may correspond to two different functions, it will probably be necessary to discuss completely the domains of convergence and of summability of (3). Such a study would be a natural complement to the present investigation, for although we have been able, as in Theorems XI and XII, to find interesting properties of the solutions of $A\phi(z) = 0$ by qualitative methods, the solutions would probably present themselves in practice through their developments (3), and it would then be desirable to have knowledge as to the function or functions defined by a given development.

Given a series (3), its theory could be made to depend on that of the convergence of a series (11), for, as we have already seen, we can obtain (11)

directly from (3). We would find thus a series (11) which converges absolutely in the entire domain of existence of any function $\phi(z)$ which may give rise to the series (3). Also, it is seen that if the series (11) thus obtained converges uniformly in an area, it converges to a function of which the original series (3) is a formal development; *the series* (11) *must therefore converge absolutely in any area in which it converges uniformly.**

We shall probably devote a separate paper to these questions.

11. **Application to the theory of analytic prolongation.** From Theorems XI and XII, it follows immediately that any function which can be represented in an area in its domain of existence by a uniformly convergent series (3), is uniform and has no isolated singularities. The importance of this fact is revealed when we consider a simple type of series (3). Assuming that the numbers a_n are positive integers, ordered according to increasing magnitude, consider the series

$$\sum_{n=1}^{\infty} c_n e^{a_n z}, \tag{12}$$

and with it, the power series

$$\sum_{n=1}^{\infty} c_n x^{a_n}. \tag{13}$$

If (13) has a radius of convergence ρ, (12) will be convergent when the real part of z is less than $\log \rho$ and will be divergent when the real part of z is greater than $\log \rho$. If (13) can be prolonged beyond its circle of convergence, (12) can also be prolonged into its field of divergence. This will be impossible if the development (3) of every solution of $A\phi(z) = 0$ converges in the entire domain of existence of the solution. We have thus the result:

If $\sum 1/|a_n|$ is convergent, and if there exists an integer r such that, for $n > r$,

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 + \frac{k}{n},$$

where k is any positive number, the series

$$\sum_{n=1}^{\infty} c_n x^{a_n}$$

has its circle of convergence as a natural boundary.

If a series (13) existed which could be prolonged beyond its circle of convergence, it could be shown that the function thus obtained would be uniform and would have no isolated singularities. Also, we would be able to sum the series throughout the domain over which it could be prolonged,

* Perhaps it is well to add, in this connection, that it can be shown as in Theorem XIII that the development (3) converges uniformly in any closed and bounded domain interior to an area in which $|\sum_1^n u_\nu|$ stays bounded as n increases, even should that area be exterior to the domain of existence of the solution which gave rise to the development.

according to the method of § 9. There is, however, no possibility of such prolongation. Indeed, Fabry* has given the following sufficient condition for the circle of convergence to be a natural boundary, which can be shown to be satisfied whenever $\sum 1/|a_n|$ is convergent:

If it is possible to choose a sequence of subscripts m such that

$$\lim_{m \doteq \infty} \sqrt[a_m]{|c_m|} = \text{lim. sup.} \sqrt[a_n]{|c_n|}$$

and such that the ratio to a_m of the number of terms with exponents between $a_m(1-\lambda)$ and $a_m(1+\lambda)$, where λ is an arbitrarily small positive number, goes to zero as m increases, the circle of convergence of (13) *is a natural boundary.*

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* E. Fabry, Acta Mathematica, vol. 22 (1898), p. 86. See also G. Faber, Muenchener Berichte, vol. 34 (1904).

ON THE EXPRESSIBILITY OF A UNIFORM FUNCTION OF SEVERAL COMPLEX VARIABLES AS THE QUOTIENT OF TWO FUNCTIONS OF ENTIRE CHARACTER*

BY

T. H. GRONWALL

1. Introduction

It is a classical fact in the theory of functions of one complex variable that any meromorphic function may be expressed as the quotient of two entire functions without common zeros. When $f(x)$ is a uniform function with essential singularities at finite distance, this theorem may be extended, as was shown by Weierstrass† for a finite number of essential singularities, and by Mittag-Leffler in the general case: $f(x)$ is expressible as the quotient of two functions of entire character (that is, uniform and without poles, but generally both having the same essential singularities as $f(x)$) without common zeros.

Before taking up the corresponding question for several variables, it is convenient to recall the following definitions:

The complex variables $x_1, x_2, \cdots, x_n$ are interior to the region $(S_1, S_2, \cdots, S_n)$ when x_1 is interior to the region S_1 in the x_1-plane, $\cdots$, x_n interior to the region S_n in the x_n-plane; the regions $S_1, \cdots, S_n$ may be simply or multiply connected.

A *uniform* function $f(x_1, x_2, \cdots, x_n)$ of the complex variables $x_1, x_2, \cdots, x_n$ is *meromorphic* in $(S_1, S_2, \cdots, S_n)$ when, in the vicinity of every point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$, we have

$$f(x_1, x_2, \cdots, x_n) = \frac{P_1(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n)}{P_0(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n)},$$

where P_0 and P_1 are power series in $x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n$. A *uniform* function $G(x_1, x_2, \cdots, x_n)$ is *of entire character* in $(S_1, S_2, \cdots, S_n)$

* Presented to the Society, October 25, 1913.

† K. Weierstrass, *Zur Theorie der eindeutigen analytischen Functionen, Mathematische Werke*, vol. 2 (Berlin, 1895), pp. 77–124. G. Mittag-Leffler, *Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante*, Acta Mathematica, vol. 4 (1884), pp. 1–79.

when holomorphic at every point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$. Two functions *of entire character* $G_0(x_1, x_2, \cdots, x_n)$ and $G_1(x_1, x_2, \cdots, x_n)$ have *a common divisor* when there exists a point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$ such that in its vicinity

$$G_0(x_1, x_2, \cdots, x_n) = P(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n) \cdot P_0(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n),$$

$$G_1(x_1, x_2, \cdots, x_n) = P(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n) \cdot P_1(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n),$$

with $P(0, 0, \cdots, 0) = 0$. Two functions *of entire character* are relatively prime when they have no common divisor.

Poincaré has shown,* by the theory of harmonic functions of four real variables, that when $n = 2$ and S_1 and S_2 contain all points at finite distance in the x_1- and x_2-planes respectively, every meromorphic function is expressible as the quotient of two entire functions without common divisor. In a later paper,† he has modified this method and extended it to n variables.

The Cauchy integral was used by Cousin‡ to prove Poincaré's result and extend it to more general regions. His most general results are the following, of which **A** may be regarded as the extension to several variables of Mittag-Leffler's theorem, while **B** generalizes Weierstrass's theorem on the existence of uniform functions with given zeros:

A. When for every point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$ there are given

(1) a region $\Gamma_{a_1, a_2, \cdots, a_n}$ consisting of n circles $|x_\nu - a_\nu| < r_\nu$ $(\nu = 1, 2, \cdots, n)$, each of these circles being interior to the corresponding region S_ν;

(2) a function $f_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n)$ uniform in $\Gamma_{a_1, a_2, \cdots, a_n}$ and such that when two regions $\Gamma_{a_1, a_2, \cdots, a_n}$ and $\Gamma_{a_1', a_2', \cdots, a_n'}$ have a region in common, the difference

$$f_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n) - f_{a_1', a_2', \cdots, a_n'}(x_1, x_2, \cdots, x_n)$$

is holomorphic in the common region;

Then there exists a function $F(x_1, x_2, \cdots, x_n)$ uniform in $(S_1, S_2, \cdots, S_n)$ and such that for every interior point $a_1, a_2, \cdots, a_n$ the difference

$$F(x_1, x_2, \cdots, x_n) - f_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n)$$

is holomorphic in $\Gamma_{a_1, a_2, \cdots, a_n}$.

* H. Poincaré, *Sur les fonctions de deux variables*, Acta Mathematica, vol. 2 (1883), pp. 97–113.

† H. Poincaré, *Sur les propriétés du potentiel et sur les fonctions Abéliennes*, Acta Mathematica, vol. 22 (1899), pp. 89–178.

‡ P. Cousin, *Sur les fonctions de n variables complexes*, Acta Mathematica, vol. 19 (1895), pp. 1–62.

B. When for every point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$ there are given

(1) a region $\Gamma_{a_1, a_2, \cdots, a_n}$ as in **A**;

(2) a function $u_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n)$ of entire character in $\Gamma_{a_1, a_2, \cdots, a_n}$ and such that when two regions $\Gamma_{a_1, a_2, \cdots, a_n}$ and $\Gamma_{a'_1, a'_2, \cdots, a'}$ have a region in common, the quotient

$$u_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n)/u_{a'_1, a'_2, \cdots, a'_n}(x_1, x_2, \cdots, x_n)$$

is holomorphic and different from zero in the common region;

Then there exists a function $G(x_1, x_2, \cdots, x_n)$ of entire character in $(S_1, S_2, \cdots, S_n)$ such that for every interior point $a_1, a_2, \cdots, a_n$ the quotient $G(x_1, x_2, \cdots, x_n)/u_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n)$ is holomorphic and different from zero in $\Gamma_{a_1, a_2, \cdots, a_n}$.

C. When a function $f(x_1, x_2, \cdots, x_n)$ is meromorphic in $(S_1, S_2, \cdots, S_n)$, it may be expressed as the quotient of two relatively prime functions of entire character* in $(S_1, S_2, \cdots, S_n)$:

$$f(x_1, x_2, \cdots, x_n) = \frac{G_1(x_1, x_2, \cdots, x_n)}{G_0(x_1, x_2, \cdots, x_n)}.$$

Cousin establishes Theorem **A** in its various stages in an entirely rigorous manner, but his proofs of Theorem **B** (and hence of Theorem **C**, which is a quite elementary consequence of **B**—see Cousin, l. c., §§ 15, 19, and 25) contain a gap (at stages α and β) which considerably restricts the regions $(S_1, S_2, \cdots, S_n)$ in which they are applicable.

In § 2, the nature of this gap is explained, and Cousin's proofs of **B** are shown to be valid when all, or all but one, of the n regions $S_1, S_2, \cdots, S_n$ are simply connected. On the other hand, it is established by an example that Cousin's construction of $G(x_1, x_2, \cdots, x_n)$ does not always yield a uniform function when two of the regions $S_1, S_2, \cdots, S_n$ are multiply connected.

The question now arises as to the validity of Theorems **B** and **C** in the cases where Cousin's proofs do not apply. In § 3 it is shown by an example that Theorem **C** is false (and consequently Theorem **B**, since **C** would follow from **B**) when two of the regions $S_1, S_2, \cdots, S_n$ are multiply connected, that is, in the very cases where Cousin's proofs fail.

* In his proofs, Cousin proceeds by four stages: first the theorems are derived for any region $(s_1, s_2, \cdots, s_n)$ interior to $(S_1, S_2, \cdots, S_n)$, and this separately for $n = 2$ (stage α) and n general (stage β). Second, a limiting process is used to extend the region of validity of the theorems from $(s_1, s_2, \cdots, s_n)$ to $(S_1, S_2, \cdots, S_n)$, and this separately when all S_ν are circles (stage γ) and when S_ν are quite general (stage δ). For convenient reference, the numbers of Cousin's theorems corresponding to Theorems **A**, **B**, and **C** of the text at the various stages are given below:

	α	β	γ	δ
A	I	IV	VII, p. 33	XI
B	III	VI	IX	XII
C	—	VII, p. 32	X	XIV

Thus the results of the present paper may be summarized in the statement that

Theorems **B** *and* **C** *are valid when, and only when,* $n - 1$ *of the* n *regions* $S_1, S_2, \cdots, S_n$ *are simply connected; the remaining region may be simply or multiply connected.*

The author wishes to acknowledge his indebtedness to Professor Osgood, to whom he communicated the example of § 3 in June, 1913, for material assistance in locating the gap in Cousin's proofs.

2. The domain of validity of Cousin's proofs of Theorem B

To abridge the notation, we shall write x for the system of $n - 1$ variables $x_1, x_2, \cdots, x_{n-1}$ and S for $(S_1, S_2, \cdots, S_{n-1})$; x_n will be denoted by y and S_n by S'. A simply connected part Σ of S we define as a system of regions $(\Sigma_1, \Sigma_2, \cdots, \Sigma_{n-1})$ where, for $\nu = 1, 2, \cdots, n - 1$, every interior or boundary point of the simply connected region Σ_ν is interior to or on the boundary of S_ν. The boundaries of $S_1, S_2, \cdots, S_{n-1}, \Sigma_1, \Sigma_2, \cdots, \Sigma_{n-1}$, and S' are assumed to be regular, that is, each is to consist of a finite number of pieces of analytic curves without singular points.

We now assume S' to be subdivided, by a finite number of pieces of regular curves, into a finite number of simply connected regions $R_1, R_2, \cdots, R_p, \cdots$. When R_n and R_p are adjacent regions, we denote by l_{np} their common boundary, or, should this consist of several pieces, any one of these. If any l_{np} is a closed curve, we cut it at three points, thus obtaining three pieces such that no two of them taken together form a closed curve. The direction of l_{np} is that which leaves the interior of the region R_n to the left, so that l_{np} and l_{pn} are the same curve described in opposite directions. Finally, let T_{np} consist of all points in the y-plane interior to at least one circle with center on l_{np} and sufficiently small radius r, this r being constant not only for different points on l_{np}, but also for all the various curves l_{np}.

The proof of Theorem **B** now depends on the following lemma:

Let a function $u_p(x, y)$ *be given for every region* R_p, *uniform and holomorphic in* (S, R_p), *boundaries included, and such that for any two adjacent regions* R_n *and* R_p, *the quotient*

$$\frac{u_p(x, y)}{u_n(x, y)} = g_{np}(x, y)$$

is holomorphic and different from zero in (S, T_{np}). *Then there exists a function* $G(x, y)$ *holomorphic in* (S, S'), *uniform in* (Σ, S'), *where* Σ *is any simply connected part of* S, *and such that in* (S, R_p) *(boundaries included, except those* y *which are end points of an* l_{np} *and lie on the boundary of* S'*) the*

quotient

$$\frac{G(x, y)}{u_p(x, y)}$$

is holomorphic and different from zero.

When S is simply connected, we may evidently let Σ coincide with S. In his formulation of the lemma (l. c., § 7; proof in § 6) Cousin makes no distinction between Σ and S, so that, when S is multiply connected (that is, one at least of $S_1, S_2, \cdots, S_{\nu-1}$ is multiply connected) he tacitly assumes the function $G(x, y)$ to be uniform in (S, S'), while the uniformity is proved only in (Σ, S').

This constitutes the gap in Cousin's proofs referred to in the introduction. It might also be objected to his proof of the lemma (l. c., § 6) that he operates throughout with the multiform functions $\log u_p(x, y)$ and their differences $\log u_p(x, y) - \log u_n(x, y)$, and that it is not quite clear what branches of these functions are meant at the various points of (S, S'); but this objection is met by a modification of Cousin's argument due to Osgood.*

Since $u_p(x, y)$ and $u_n(x, y)$ are uniform in (S, T_{np}) by hypothesis, and their quotient $g_{np}(x, y)$ is holomorphic and different from zero in the same region, it follows that writing

$$G_{np}(x, y) = \log g_{np}(x, y),$$

where that branch of $\log g_{np}(x, y)$ is taken which assumes its principal value at some point x_0, y_0 interior to (Σ, T_{np}), the function $G_{np}(x, y)$ is holomorphic in (S, T_{np}) and uniform in (Σ, T_{np}). Next let

$$I_{np}(x, y) = \frac{1}{2\pi i}\int_{l_{np}}^{*} \frac{G_{np}(x, z)\,dz}{z - y},$$

the integral being taken in the positive direction of l_{np}. This function is holomorphic for all y at finite or infinite distance, except those on the curve l_{np}, and for any x in S, and uniform for the same y and any x in Σ. Moreover, as shown in Cousin §§ 2–3,

$$I_{np}(x, y) = H(x, y) + G_{np}(x, y)\lambda_{np}(y),$$

$$\lambda_{np}(y) = \frac{1}{2\pi i}\log\frac{y - b}{y - a}, \qquad \lambda_{np}(\infty) = 0,$$

where a and b are the end points of l_{np}, $\log[(y - b)/(y - a)]$ is that branch of the logarithm which vanishes for $y = \infty$, so that $\lambda_{np}(y)$ is uniform and holomorphic in the whole y-plane except on the curve l_{np}, and finally $H(x, y)$

* Letter to the author, July 7, 1913. This modified proof is reproduced here with the permission of Professor Osgood.

is holomorphic in (S, T_{np}) and uniform in (Σ, T_{np}). Now write

$$\Phi(x, y) = \sum I_{np}(x, y),$$

where the summation is extended over all the curves l_{np} which are common to the boundaries of two regions R (each curve taken once, and not in the two subscript combinations l_{np} and l_{pn}), and define

$$\phi_n(x, y) = \Phi(x, y) \text{ in } (S, R_n).$$

Then $\phi_n(x, y)$ is holomorphic in (S, R_n) and uniform in (Σ, R_n), boundaries included except the end points of the various l_{np} belonging to the boundary of R_n. Denoting by $\phi_n(x, y)_p$ the analytic continuation of $\phi_n(x, y)$ when x describes any path in S and y a path in T_{np} starting at a point inside R_n and ending at a point inside R_p, but not passing through an end point of l_{np}, we have (Cousin, l. c., §§ 2–3)

$$\phi_n(x, y)_p = \phi_p(x, y) + G_{np}(x, y). \tag{1}$$

A point $y = b$ interior to S' is called a *vertex* when it is an end point of any l_{np}.

Now make

$$\overline{G}_n(x, y) = u_n(x, y)e^{\phi_n(x, y)} \text{ in } (S, R_n);$$

then it follows from (1) that $\overline{G}_p(x, y)$ is the analytic continuation of $\overline{G}_n(x, y)$ across l_{np} (the path in the y-plane leading from R_n into R_p not crossing l_{np} at a vertex), and consequently the continuation of $\overline{G}_n(x, y)$ along a closed path in the y-plane not passing through any vertex brings us back to $\overline{G}_n(x, y)$. We may therefore define a single function $\overline{G}(x, y)$ by the consistent conditions $\overline{G}(x, y) = \overline{G}_n(x, y)$ in (S, R_n), and this $\overline{G}(x, y)$ is visibly uniform in (Σ, S'). Moreover, the quotient $\overline{G}(x, y)/u_p(x, y)$ is holomorphic and different from zero in (S, R_p), boundaries included, except when y coincides with an end point of an l_{np} while x takes any value inside or on the boundary of S.

We shall now modify $\overline{G}(x, y)$ so as to remove the last restriction for those end points of an l_{np} which are vertices. Let b be a vertex, and suppose that, for instance, $R_1, R_2, \cdots, R_m$ are those regions R which are adjacent to this vertex. Let $1 \leqq \nu \leqq m$ and denote by R'_ν that part of R_ν which lies within or on the circle $|y - b| = r'$, where r' is less than the radius r of the circles used in defining all T_{np}. Then we have in (S, R'_ν)

$$\begin{aligned}\phi_\nu(x, y) = \Phi(x, y) = A(x, y) + G_{12}(x, y)\lambda_{12}(y) + G_{23}(x, y)\lambda_{23}(y) \\ + \cdots + G_{m-1, m}(x, y)\lambda_{m-1, m}(y) + G_{m1}(x, y)\lambda_{m1}(y),\end{aligned}$$

$A(x, y)$ being holomorphic in $(S, |y - b| \leqq r')$ and uniform in $(\Sigma, |y - b| \leqq r')$. Make

$$L_\nu(y - b) = \frac{1}{2\pi i}\log(y - b),$$

where any branch of the logarithm is chosen and rendered uniform by a cut issuing from $y = b$, but having no other point in common with R'_ν or its boundary. None of the l_{np} abutting at b being closed, we may continue $\lambda_{np}(y)$ analytically from $y = \infty$ to a point inside R'_ν along a curve intersecting none of these l_{np}, and in the relation

$$\lambda_{np}(y) - L_\nu(y - b) = -\frac{1}{2\pi i}\log(y - a),$$

where now $\log(y - a)$ is a definite branch of the logarithm, for y in R'_ν, the right-hand member is holomorphic in the entire region $|y - b| \leqq r'$. Hence we have, *for y interior to R'_ν*,

$$\phi_\nu(x, y) = B_\nu(x, y) + [G_{12}(x, y) + G_{23}(x, y) + \cdots$$
$$+ G_{m-1,m}(x, y) + G_{m1}(x, y)]\, L_\nu(y - b),$$

where $B_\nu(x, y)$ is holomorphic in $(S, |y - b| \leqq r')$ and uniform in $(\Sigma, |y - b| \leqq r')$. On the other hand, the sum in brackets equals

$$\log\frac{u_2(x, y)}{u_1(x, y)} + \log\frac{u_3(x, y)}{u_2(x, y)} + \cdots + \log\frac{u_m(x, y)}{u_{m-1}(x, y)} + \log\frac{u_1(x, y)}{u_m(x, y)},$$

where each log refers to a definite branch of the function—the branch chosen at the beginning, and this sum therefore equals a definite value of $\log 1$, which we denote by $2\pi i K_b$, the integer K_b being evidently independent of ν. Consequently, for y interior to R'_ν,

$$(y - b)^{-K_b}\,\overline{G}(x, y) = u_\nu(x, y)\, e^{\phi_\nu(x, y) - 2\pi i K_b L_\nu(y - b)},$$

or

$$(y - b)^{-K_b}\,\overline{G}(x, y) = u_\nu(x, y)\, e^{B_\nu(x, y)};$$

but the expression to the right being holomorphic in $(S, |y - b| \leqq r')$ and uniform in $(\Sigma, |y - b| \leqq r')$, it follows by analytic continuation that the same is true of the left-hand member, and that the quotient of the latter by $u_\nu(x, y)$, which equals $e^{B_\nu(x, y)}$ in (S, R'_ν), is holomorphic and different from zero in that region.

Finally determine the integer K_b for each vertex b and write

$$G(x, y) = \overline{G}(x, y)\prod_b (y - b)^{-K_b},$$

the product extending over all vertices. It then follows immediately from the preceding argument that $G(x, y)$ has all the properties mentioned in the lemma.

As already stated, Cousin tacitly assumes that from the proven uniformity of $G(x, y)$ in (Σ, S') it follows that $G(x, y)$ is also uniform in $(S,')$ S when S is multiply connected.

I shall now show by an example that this conclusion is not legitimate; it is evidently sufficient to assume $n = 2$, so that now x stands for a single variable, and S for a region in the x-plane. This example, as well as the one in § 3, is based on the simplest properties of Theta functions of two variables. It is well known that, given the constants τ_{11}, τ_{12}, τ_{22} such that the real part of $2\pi i(\tau_{11} n_1^2 + 2\tau_{12} n_1 n_2 + \tau_{22} n_2^2)$ is a definite negative quadratic form in n_1 and n_2, the two expressions*

$$\phi_\nu(v_1, v_2) = \sum_{n_1, n_2 = -\infty}^{+\infty} \operatorname{Exp}\left[\left(n_1 - \frac{\nu}{2}\right)^2 \tau_{11} + 2\left(n_1 - \frac{\nu}{2}\right) n_2 \tau_{12} + n_2^2 \tau_{22} - 2\left(n_1 - \frac{\nu}{2}\right) v_1 - 2n_2 v_2\right], \tag{2}$$

where $\nu = 0$ or 1, define entire functions of v_1 and v_2 with the properties

$$\begin{aligned} \phi_\nu(v_1 + 1, v_2) &= \phi_\nu(v_1, v_2), \\ \phi_\nu(v_1, v_2 + \tfrac{1}{2}) &= \phi_\nu(v_1, v_2), \\ \phi_\nu(v_1 + \tau_{11}, v_2 + \tau_{12}) &= \operatorname{Exp}(-2v_1 - \tau_{11}) \cdot \phi_\nu(v_1, v_2), \\ \phi_\nu(v_1 + \tau_{12}, v_2 + \tau_{22}) &= \operatorname{Exp}(-2v_2 - \tau_{22}) \cdot \phi_\nu(v_1, v_2) \end{aligned} \qquad (\nu = 0, 1).$$

Assume $\tau_{12} \neq 0$, introduce new variables w_1 and w_2 by the relations

$$\tau_{12} w_1 = -2\tau_{22} v_1 + 2\tau_{12} v_2, \qquad \tau_{12} w_2 = v_1,$$

and write $\phi_\nu(v_1, v_2) = \psi_\nu(w_1, w_2)$; then $\psi_\nu(w_1, w_2)$ are entire functions of w_1 and w_2 with the properties

$$\begin{aligned} \psi_\nu(w_1 + 1, w_2) &= \psi_\nu(w_1, w_2), \\ \psi_\nu(w_1, w_2 + 1) &= \operatorname{Exp}(-w_1 - 2\tau_{22} w_2 - \tau_{22}) \cdot \psi_\nu(w_1, w_2), \\ \psi_\nu\left(w_1 - \frac{2\tau_{22}}{\tau_{12}}, w_2 + \frac{1}{\tau_{12}}\right) &= \psi_\nu(w_1, w_2), \qquad (\nu = 0, 1). \\ \psi_\nu\left(w_1 + \frac{2\tau_{12}^2 - 2\tau_{11}\tau_{22}}{\tau_{12}}, w_2 + \frac{\tau_{11}}{\tau_{12}}\right) &= \operatorname{Exp}(-2\tau_{12} w_2 - \tau_{11}) \cdot \psi_\nu(w_1, w_2) \end{aligned} \tag{3}$$

Finally write $\psi(w_1, w_2) = \operatorname{Exp}(\tau_{22} w_2^2) \cdot \psi_0(w_1, w_2)$; then the entire function $\psi(w_1, w_2)$ has the properties

$$\begin{aligned} \psi(w_1 + 1, w_2) &= \psi(w_1, w_2), \\ \psi(w_1, w_2 + 1) &= \operatorname{Exp}(-w_1) \cdot \psi(w_1, w_2). \end{aligned} \tag{4}$$

* To simplify the typography, we shall use the notation $e^{2\pi i x} = \operatorname{Exp}(x)$.

Once more we introduce new variables by the equations

$$x = \operatorname{Exp}(w_1), \qquad y = \operatorname{Exp}(w_2) \tag{5}$$

and write

$$u(x, y) = \psi(w_1, w_2) = \psi\left(\frac{1}{2\pi i}\log x, \frac{1}{2\pi i}\log y\right); \tag{6}$$

then $u(x, y)$ is holomorphic for all x, y at finite distance, except $x = 0$, $y = y$ and $x = x$, $y = 0$. Starting with some definite branches of $\log x$ and $\log y$, say those that equal zero for $x = 1$ and $y = 1$ respectively, it follows from (4) that $u(x, y)$ is uniform in respect to x, while the analytic continuation along a path winding about $y = 0$ once in the positive sense transforms the initial branch $u(x, y)$ into a new branch $\bar{u}(x, y)$ such that

$$\bar{u}(x, y) = \frac{1}{x}u(x, y). \tag{7}$$

Now let us construct the function $G(x, y)$ of the lemma from the following data:

S: the circular ring $\frac{1}{2} < |x| < 2$;
S': the circular ring $\frac{1}{2} < |y| < 2$;
R_1: the part of S' to the right of the imaginary axis;
R_2: the part of S' to the left of the imaginary axis;
l_{12}: the straight line segment from $y = 2i$ to $y = \frac{1}{2}i$;
l'_{12}: the straight line segment from $y = -\frac{1}{2}i$ to $y = -2i$, so that the common part of the boundaries of R_1 and R_2 consists of l_{12} and l'_{12};
$u_1(x, y)$: the initial branch of $u(x, y)$ defined above;
$u_2(x, y)$: the analytic continuation of $u_1(x, y)$ across the line l_{12}.

Then $u_1(x, y)$ and $u_2(x, y)$ are uniform and holomorphic in (S, R_1) and (S, R_2) respectively, boundaries included. On l_{12},

$$g_{12}(x, y) = \frac{u_2(x, y)}{u_1(x, y)} = 1,$$

while on l'_{12} we have

$$g'_{12}(x, y) = \frac{u_2(x, y)}{u_1(x, y)} = \frac{1}{x}$$

according to (7). We now make

$$G_{12}(x, y) = \log 1 = 0, \qquad G'_{12}(x, y) = -\log x,$$

where that branch of the logarithm is taken which vanishes at $x = 1$; since there are no vertices and therefore no integers K_b to be determined, we may proceed at once to write down $\Phi(x, y)$:

$$\Phi(x, y) = \frac{1}{2\pi i}\int_{-\frac{1}{2}i}^{-2i} \frac{-\log x dz}{z - y} = \frac{1}{2\pi i}\log x \cdot \log\frac{y + \frac{1}{2}i}{y + 2i},$$

where the last logarithm is the branch that vanishes for y infinite. Finally we obtain

$$G(x, y) = u_p(x, y) \operatorname{Exp}\left(\frac{1}{2\pi i} \cdot \frac{1}{2\pi i} \log x \cdot \log \frac{y + \frac{1}{2}i}{y + 2i}\right) \tag{8}$$

in (S, R_p) for $p = 1, 2$. This $G(x, y)$ now has all the properties indicated in the lemma (as is also readily verified directly in this particular case). Nevertheless, $G(x, y)$ is not uniform in (S, S'), for letting x describe a closed path in S starting and ending at $x = 1$, and winding about $x = 0$ once in the positive sense, while y describes a closed path interior to R_1, $\log x$ increases by $2\pi i$, while $\log (y + \frac{1}{2}i)/(y + 2i)$ and $u_1(x, y)$ remain unchanged, and we arrive at a branch $\bar{G}(x, y)$ connected with the initial branch $G(x, y)$ by the relation

$$\bar{G}(x, y) = \frac{y + \frac{1}{2}i}{y + 2i} G(x, y).$$

Hence Cousin's lemma, and with it his proofs of Theorem **B**, are valid when, and only when, not more than one of the regions $S_1, S_2, \cdots, S_n$ is multiply connected.

3. Example of a function of two variables, meromorphic in a region (S, S'), which cannot be expressed as the quotient of two relatively prime functions of entire character

From (3) it is evident that the quotient

$$\frac{\psi_1(w_1, w_2)}{\psi_0(w_1, w_2)} = \frac{\phi_1(v_1, v_2)}{\phi_0(v_1, v_2)}$$

is a meromorphic quadruply periodic function of w_1 and w_2 with the periods

$$\begin{array}{lllll} 1, & 0, & -\dfrac{2\tau_{22}}{\tau_{12}}, & \dfrac{2\tau_{12}^2 - 2\tau_{11}\tau_{22}}{\tau_{12}} & \text{in } w_1, \\[2ex] 0, & 1, & \dfrac{1}{\tau_{12}}, & \dfrac{\tau_{11}}{\tau_{12}} & \text{in } w_2. \end{array}$$

By (2), $\phi_0(v_1, v_2)$ contains only even, and $\phi_1(v_1, v_2)$ only odd, powers of Exp. (v_1); hence these two functions are linearly independent, and the quotient considered is not a constant. Introducing the variables x and y by (5) and writing

$$f(x, y) = \frac{\psi_1(w_1, w_2)}{\psi_0(w_1, w_2)},$$

$f(x, y)$ is a non-constant, uniform function of x and y, meromorphic in the region (S, S'), where S consists of all points at finite distance in the x-plane,

the point $x = 0$ excepted, and S' is defined similarly in the y-plane. This function has the properties

$$(9)\qquad \begin{aligned} f(hx, ky) &= f(x, y), \\ f(lx, my) &= f(x, y), \end{aligned}$$

where

$$(10)\qquad \begin{aligned} h &= \operatorname{Exp}\left(-\frac{2\tau_{22}}{\tau_{12}}\right), \qquad k = \operatorname{Exp}\left(\frac{1}{\tau_{12}}\right), \\ l &= \operatorname{Exp}\left(\frac{2\tau_{12}^2 - 2\tau_{11}\tau_{22}}{\tau_{12}}\right), \qquad m = \operatorname{Exp}\left(\frac{\tau_{11}}{\tau_{12}}\right). \end{aligned}$$

Now let us subject $\tau_{11}, \tau_{12}, \tau_{22}$ to the further condition that

$$(11)\qquad l^a m^b \neq h^c k^d$$

for any integers a, b, c, and d which are not all equal to zero. By (10), this is equivalent to the condition that the equation

$$(12)\qquad b\tau_{11} + n\tau_{12} + 2c\tau_{22} + 2a(\tau_{12}^2 - \tau_{11}\tau_{22}) - d = 0$$

shall have no solution in integers a, b, c, d, n which are not all equal to zero.* Then $f(x, y)$ cannot be expressed as the quotient of two relatively prime functions of entire character in (S, S'). For the purpose of an example; it is sufficient to carry out the proof in a special case, giving numerical values to $\tau_{11}, \tau_{12}, \tau_{22}$.† Let us make

$$\tau_{11} = i, \qquad \tau_{12} = \frac{1}{\sqrt[4]{2}}, \qquad \tau_{22} = i\sqrt{2};$$

then the real part of $2\pi i(\tau_{11} n_1^2 + 2\tau_{12} n_1 n_2 + \tau_{22} n_2^2)$ is $-2\pi(n_1^2 + \sqrt{2}n_2^2)$, a definite negative quadratic form in n_1 and n_2. Furthermore $\tau_{12} \neq 0$, and (12) gives upon separation of the real and imaginary parts

$$b + 2c\sqrt{2} = 0, \qquad n + 3a\sqrt[4]{8} - d\sqrt[4]{2} = 0,$$

whence

$$b = c = 0, \qquad n^2 + 12ad - (18a^2 + d^2)\sqrt{2} = 0, \qquad a = d = n = 0.$$

Hence (11) is satisfied, and in particular we have for any integers λ and μ, except $\lambda = \mu = 0$,

$$(13)\qquad h^\lambda k^\mu - 1 \neq 0, \qquad l^\lambda m^\mu - 1 \neq 0.$$

* In the theory of Theta functions, this condition expresses the fact that the period system $\tau_{11}, \tau_{12}, \tau_{22}$ is non-singular.

† This has the advantage of simplifying the convergence proof for the series (19).

Now assume that $f(x, y)$ can be expressed in the form*

$$f(x, y) = \frac{G_1(x, y)}{G_0(x, y)}, \tag{14}$$

where $G_0(x, y)$ and $G_1(x, y)$ are of entire character and relatively prime in (S, S'); we shall show that this leads to a contradiction. From (9) and (14) it follows that

$$\frac{G_0(hx, ky)}{G_0(x, y)} = \frac{G_1(hx, ky)}{G_1(x, y)}, \qquad \frac{G_0(lx, my)}{G_0(x, y)} = \frac{G_1(lx, my)}{G_1(x, y)},$$

and since $G_0(x, y)$ and $G_1(x, y)$ are relatively prime, we conclude that both these quotients, which are evidently uniform, are holomorphic and different from zero in (S, S').† Let us denote them by $g(x, y)$ and $g'(x, y)$ respectively; then

$$G_\nu(hx, ky) = g(x, y)G_\nu(x, y), \qquad G_\nu(lx, my) = g'(x, y)G_\nu(x, y) \qquad (\nu = 0, 1). \tag{15}$$

Since $g(x, y)$ is of entire character and different from zero in (S, S'), we may expand its logarithmic derivatives in Laurent's series‡

$$\frac{\partial \log g(x, y)}{\partial x} = \sum_{\lambda, \mu = -\infty}^{+\infty} a_{\lambda\mu}\, x^\lambda y^\mu, \qquad \frac{\partial \log g(x, y)}{\partial y} = \sum_{\lambda, \mu = -\infty}^{+\infty} b_{\lambda\mu}\, x^\lambda y^\mu,$$

both series being absolutely and uniformly convergent for $\epsilon \leqq |x| \leqq 1/\epsilon$, $\epsilon \leqq |y| \leqq 1/\epsilon$, where ϵ is as small as we please. From

$$\frac{\partial^2 \log g(x, y)}{\partial y \partial x} = \frac{\partial^2 \log g(x, y)}{\partial x \partial y}$$

it follows that

$$\sum \mu a_{\lambda\mu}\, x^\lambda y^{\mu-1} = \sum \lambda b_{\lambda\mu}\, x^{\lambda-1} y^\mu,$$

so that in particular $\mu a_{-1,\mu} = 0$, $\lambda b_{\lambda,-1} = 0$, whence

$$a_{-1,0} = a, \qquad a_{-1,\mu} = 0 \qquad (\mu \neq 0),$$

$$b_{0,-1} = b, \qquad b_{\lambda,-1} = 0 \qquad (\lambda \neq 0).$$

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* The following investigation is closely related to one made by Appell to an entirely different purpose in his paper *Sur les fonctions périodiques de deux variables*, Journal de Mathématiques, ser. 4, vol. 7 (1891), pp. 157–219. See pp. 185–201.

† This is a simple consequence of Weierstrass' preparation theorem; compare Cousin, l. c., § 15, and Appell, l. c., pp. 182–185.

‡ K. Weierstrass, *Einige auf die Theorie der analytischen Funktionen mehrerer Veränderlichen sich beziehende Sätze, Mathematische Werke*, vol. 2 (Berlin, 1895), pp. 135–188. See pp. 183–188.

Treating $g'(x, y)$ in the same way, and integrating, we finally obtain

$$(16)\qquad \begin{aligned} g(x, y) &= x^a y^b \operatorname{Exp}\left(\sum_{\lambda,\mu=-\infty}^{+\infty} A_{\lambda\mu} x^\lambda y^\mu \right), \\ g'(x, y) &= x^c y^d \operatorname{Exp}\left(\sum_{\lambda,\mu=-\infty}^{+\infty} B_{\lambda\mu} x^\lambda y^\mu \right), \end{aligned}$$

the series being absolutely and uniformly convergent as before, and from the uniformity of $g(x, y)$ and $g'(x, y)$ it is evident that a, b, c, d are all integers. We arrive at a relation between $g(x, y)$ and $g'(x, y)$ by observing that according to (15)

$$\frac{G_\nu(hlx, kmy)}{G_\nu(x, y)} = \frac{G_\nu(hlx, kmy)}{G_\nu(lx, my)} \cdot \frac{G_\nu(lx, my)}{G_\nu(x, y)} = g(lx, my)\, g'(x, y),$$

$$\frac{G_\nu(lhx, mky)}{G_\nu(x, y)} = \frac{G_\nu(lhx, mky)}{G_\nu(hx, ky)} \cdot \frac{G_\nu(hx, ky)}{G_\nu(x, y)} = g'(hx, ky)\, g(x, y),$$

whence

$$g(lx, my)\, g'(x, y) = g'(hx, ky)\, g(x, y).$$

Introducing the expressions (16) into this relation, we obtain

$$l^a m^b \operatorname{Exp}\left[\sum (A_{\lambda\mu} l^\lambda m^\mu + B_{\lambda\mu}) x^\lambda y^\mu \right] = h^c k^d \operatorname{Exp}\left[\sum (B_{\lambda\mu} h^\lambda k^\mu + A_{\lambda\mu}) x^\lambda y^\mu \right],$$

which evidently gives

$$(17)\qquad A_{\lambda\mu}(l^\lambda m^\mu - 1) = B_{\lambda\mu}(h^\lambda k^\mu - 1)$$

and $l^a m^b = h^c k^d$. But in the last relation it follows from (11)—and this is the main point of the proof—that the integers $a, b; c$, and d are all equal to zero. Moreover, (13) shows that we may write (17) in the form

$$(18)\qquad \frac{A_{\lambda\mu}}{h^\lambda k^\mu - 1} = \frac{B_{\lambda\mu}}{l^\lambda m^\mu - 1}, \qquad \text{except for } \lambda = \mu = 0.$$

Denote by $\sum'$ a series from which the combination $\lambda = \mu = 0$ is excluded, and write

$$(19)\qquad G(x, y) = \sum_{\lambda,\mu=-\infty}^{+\infty}{}' \frac{A_{\lambda\mu}}{h^\lambda k^\mu - 1} x^\lambda y^\mu = \sum_{\lambda,\mu=-\infty}^{+\infty}{}' \frac{B_{\lambda\mu}}{l^\lambda m^\mu - 1} x^\lambda y^\mu;$$

then (18) shows that the two definitions of $G(x, y)$ are formally consistent. For the convergence proof, separate the terms where $\lambda \neq 0$ from those with $\lambda = 0$; we obtain with the aid of (18)

$$G(x, y) = \sum_{\mu=-\infty}^{+\infty} \sum_{\lambda\neq 0} \frac{A_{\lambda\mu}}{h^\lambda k^\mu - 1} x^\lambda y^\mu + \sum_{\mu\neq 0} \frac{B_{0\mu}}{m^\mu - 1} y^\mu .$$

Introducing the numerical values of τ_{11}, τ_{12}, τ_{22} in (10), we find

$$h = e^{4\pi\sqrt[4]{2}}, \qquad k = e^{2\pi i\sqrt[4]{2}}, \qquad m = e^{-2\pi\sqrt[4]{2}},$$

and consequently

$$|h^\lambda k^\mu - 1| \geqq \big|\,|h|^\lambda |k|^\mu - 1\big| = |e^{4\pi\sqrt[4]{2}\cdot\lambda} - 1|;$$

the last expression being greater than $e - 1$ or $1 - e^{-1}$ according as λ is a positive or negative integer, we have $|h^\lambda k^\mu - 1| > \frac{1}{2}$ for $\lambda \neq 0$, and similarly $|m^\mu - 1| > \frac{1}{2}$ for $\mu \neq 0$. Therefore (19) converges absolutely and uniformly in the same region as (16), that is, for $\epsilon \leqq |x| \leqq 1/\epsilon$, $\epsilon \leqq |y| \leqq 1/\epsilon$. Evidently $G(x, y)$ satisfies the relations

$$(20)\qquad \begin{aligned} G(hx, ky) - G(x, y) &= \textstyle\sum' A_{\lambda\mu}\, x^\lambda y^\mu, \\ G(lx, my) - G(x, y) &= \textstyle\sum' B_{\lambda\mu}\, x^\lambda y^\mu. \end{aligned}$$

If we now write

$$G'_\nu(x, y) = \operatorname{Exp}[-G(x, y)] \cdot G_\nu(x, y) \qquad (\nu = 0, 1),$$

$G'_0(x, y)$ and $G'_1(x, y)$ are of entire character (and relatively prime) in (S, S'), and by (14)

$$(21)\qquad f(x, y) = \frac{G'_1(x, y)}{G'_0(x, y)}.$$

From (15), (16), and (20) we find, bearing in mind that $a = b = c = d = 0$,

$$(22)\qquad \begin{aligned} G'_\nu(hx, ky) &= \operatorname{Exp}(A_{00}) \cdot G'_\nu(x, y), \\ G'_\nu(lx, my) &= \operatorname{Exp}(B_{00}) \cdot G'_\nu(x, y) \end{aligned} \qquad (\nu = 0, 1).$$

Expanding $G'_0(x, y)$ and $G'_1(x, y)$ in Laurent's series

$$G'_0(x, y) = \sum_{\lambda,\,\mu=-\infty}^{+\infty} C_{\lambda\mu}\, x^\lambda y^\mu, \qquad G'_1(x, y) = \sum_{\lambda,\,\mu=-\infty}^{+\infty} D_{\lambda\mu}\, x^\lambda y^\mu,$$

the first equation (22) gives

$$C_{\lambda\mu}[h^\lambda k^\mu - \operatorname{Exp}(A_{00})] = D_{\lambda\mu}[h^\lambda k^\mu - \operatorname{Exp}(A_{00})] = 0.$$

Since $G'_0(x, y)$ is not identically zero, one $C_{\lambda\mu}$ at least must be different from zero, say $C_{\rho\sigma}$, so that $h^\rho k^\sigma - \operatorname{Exp}(A_{00}) = 0$. If $h^\lambda k^\mu - \operatorname{Exp}(A_{00}) = 0$, it follows that $h^{\lambda-\rho} k^{\mu-\sigma} - 1 = 0$, whence $\lambda = \rho$, $\mu = \sigma$ by (13). Therefore $h^\lambda k^\mu - \operatorname{Exp}(A_{00}) \neq 0$, and $C_{\lambda\mu} = D_{\lambda\mu} = 0$ except for $\lambda = \rho$, $\mu = \sigma$, and (21) gives

$$f(x, y) = \frac{D_{\rho\sigma}\, x^\rho y^\sigma}{C_{\rho\sigma}\, x^\rho y^\sigma} = \text{const.}$$

But we have seen from the definition of $f(x, y)$ that this function is not a constant, and this contradiction shows that Theorem **C** (and consequently

Theorem **B**, since **B** implies **C**) is not valid when two of the regions S_1, S_2, $\cdots$, S_n are multiply connected.

It is possible however to express our function $f(x, y)$ as the quotient of two functions $G_1(x, y)$ and $G_0(x, y)$ of entire character in (S, S'), if we remove the condition that these two functions shall be relatively prime. To prove this, let $\rho = 0$ or 1 and write

$$\psi_2(w_1, w_2) = \operatorname{Exp}(2\tau_{22} w_2^2) \cdot \psi_\rho(w_1, -w_2);$$

it then follows from (3) that

$$\psi_2(w_1 + 1, w_2) = \psi_2(w_1, w_2),$$

$$\psi_2(w_1, w_2 + 1) = \operatorname{Exp}(w_1 + 2\tau_{22} w_2 + \tau_{22})\psi_2(w_1, w_2),$$

so that

$$\begin{aligned} \psi_2(w_1 + 1, w_2)\psi_\nu(w_1 + 1, w_2) &= \psi_2(w_1, w_2)\psi_\nu(w_1, w_2), \\ \psi_2(w_1, w_2 + 1)\psi_\nu(w_1, w_2 + 1) &= \psi_2(w_1, w_2)\psi_\nu(w_1, w_2) \end{aligned} \qquad (\nu = 0, 1),$$

and consequently, writing

$$G_\nu(x, y) = \psi_2(w_1, w_2)\psi_\nu(w_1, w_2) \qquad (\nu = 0, 1),$$

$G_0(x, y)$ and $G_1(x, y)$ are both *uniform* functions of x and y, holomorphic in (S, S'). Since $f(x, y) = \psi_1(w_1, w_2)/\psi_0(w_1, w_2)$, we have in

$$f(x, y) = \frac{G_1(x, y)}{G_0(x, y)}$$

a representation of $f(x, y)$ of the required character. Evidently $G_0(x, y)$ and $G_1(x, y)$ have here the common manifold of zeros defined by

$$\psi_2(w_1, w_2) = 0,$$

and from what we have proved before regarding $f(x, y)$, it follows that the common divisor cannot be removed without destroying the uniformity of $G_0(x, y)$ and $G_1(x, y)$.

In a subsequent paper, it will be shown that this representation as the quotient of two functions of entire character with common divisor is possible for any function $f(x, y)$, meromorphic everywhere at finite distance except at the points defined by $G(x, y) = 0$, where $G(x, y)$ is an entire function. The common divisor cannot in general be removed except when $G(x, y)$ is irreducible.

CERTAIN FORMAL INVARIANCES IN BOOLEAN ALGEBRAS*

BY

NORBERT WIENER

Many mathematical systems are defined through postulates concerning (1) a *class* (K) of *elements* $(a, b, c, \cdots)$, and (2) a certain definite group of *relations* or *rules of combination* joining these elements to one another.

These relations or rules of combination form only a part of those which may be said to belong to the system, for the latter may be considered to contain all those relations and rules which may be defined as logical functions of those with which the postulates deal and of certain selected elements of the system, and of nothing else. Thus, ordinary real algebra may be defined by postulates involving only the rules of combination $+$ and $\times$, but it also contains as operations the rules of combination which, when applied to x and y, yield us $x^2 + 2xy$, or $x + 1/y$, or x^y. It consequently becomes an interesting question whether the postulates of a given system deal with relations or rules of combination whose position in the system is absolutely different from that of any of the other relations or rules belonging to the system; and if this is not so, it is natural to ask what other relations or rules of the system have formal properties similar to those of the entities which form the subject-matter of the postulates; for it would seem that a system whose postulates deal with entities which occupy a unique position in it has in some sense received a more thoroughgoing analysis than one where this is not the case. As the formal properties of the relations and operations in terms of which a system is defined, in so far as they are determinate, are given in the postulates, this question reduces itself to the investigation of what relations or operations of the system satisfy the postulates. We shall discuss this question in the case of the boolean algebras, although in a somewhat limited form.

This limitation consists in a somewhat narrower definition of the statement that an operation belongs to a given boolean algebra. We shall say that an operation belongs to a boolean algebra if it is the result of the performance of any finite sequence of the operations of the algebra—logical addition, logical multiplication, and negation—on the operands and certain specified "constant" elements of the algebra. That is, if it can be represented in the

* Presented to the Society, December 27, 1916.

established symbolism of the algebra. It was pointed out by Boole* that if we represent the "logical sum" of x and y by $x + y$, their "logical product" by xy, and the negation of x by $\bar{x}$, any operation on x and y may be written in the form $Axy + Bx\bar{y} + C\bar{x}y + D\bar{x}\bar{y}$, where A, B, C, and D are any elements of the algebra you choose. E. V. Huntington has shown† that a set of postulates may be developed for a boolean algebra in terms of the relation of logical addition alone. Our investigation will hence consist in seeing what conditions the coefficients A, B, C, and D must fulfil in order that the operation $Axy + Bx\bar{y} + C\bar{x}y + D\bar{x}\bar{y}$ may satisfy the conditions expressed in these postulates.

In the paper referred to above, Huntington says: ". . . We take as the fundamental concepts a class, K, with a rule of combination, $\oplus$; and as the fundamental propositions, the following nine postulates:

"*A.* *$a \oplus a = a$ whenever a and $a \oplus a$ belong to the class.*

"*B.* *$a \oplus b = b \oplus a$ whenever a, b, $a \oplus b$, and $b \oplus a$ belong to the class.*

"*C.* *$(a \oplus b) \oplus c = a \oplus (b \oplus c)$ whenever a, b, c, $a \oplus b$, $b \oplus c$, $(a \oplus b) \oplus c$, and $a \oplus (b \oplus c)$ belong to the class.*

"*D.* *There is an element $\wedge$ such that $a \oplus \wedge = a$ for every element a.*

"*E.* *There is an element $\vee$ such that $\vee \oplus a = \vee$ for every element a.*

"*F.* *If a and b belong to the class, then $a \oplus b$ belongs to the class.*

"*G.* *If the elements $\wedge$ and $\vee$ in Postulates D and E exist and are unique, then for every element a there is an element $\bar{a}$ such that* (1°) *if $x \oplus a = a$ and $x \oplus \bar{a} = \bar{a}$, then $x = \wedge$; and* (2°) *$a \oplus \bar{a} = \vee$.*

"*H.* *If Postulates A, D, E, and G hold, and if $a \oplus \bar{b} \neq \bar{b}$, then there is an element $x \neq \wedge$ such that $a \oplus x = a$ and $b \oplus x = b$.*

"*J.* *There are at least two elements, x and y, such that $x \neq y$.*"

Huntington's $\oplus$ is meant to represent the operation of logical addition. We shall use $\oplus$ as a symbol for any operation in the algebra of logic which possesses such formal properties as to satisfy Huntington's postulates, and shall express logical addition and multiplication proper by the same symbols and according to the same conventions as those used in ordinary algebra. We shall indicate negation by a superposed bar, and the operation which takes the place of negation when $\oplus$ is substituted for $+$ by an accent.

If we take $Axy + Bx\bar{y} + C\bar{x}y + D\bar{x}\bar{y}$ as our $x \oplus y$, Huntington's postulate A becomes:

$$Aa + D\bar{a} = a.$$

As this is true independently of the value of a, we may substitute for it the universe, 1, or the null-class, 0. We thus see that $A = 1$ and that $D = 0$. The relation $x \oplus y$ can consequently be written in the form $xy + Bx\bar{y} + C\bar{x}y$.

* *Laws of Thought*, pp. 73–78.

† These Transactions, vol. 5 (1904), pp. 306–308.

Postulate B tells us that our operation is symmetrical with regard to x and y. Consequently $B = C$, and $x \oplus y$ reduces itself to the form

$$xy + B(x + y).$$

Postulate C is the law of associativity. $a \oplus (b \oplus c)$ becomes

$$a[bc + B(b + c)] + B[a + bc + B(b + c)].$$

This reduces to $abc + B(a + b + c)$. As this is a symmetrical function of a, b, and c, it is equivalent to $c \oplus (a \oplus b)$, and this, by postulate B, is but another form of $(a \oplus b) \oplus c$. Postulate C consequently imposes no restriction on an operation in the algebra of logic that is not already imposed by postulates A and B.

Postulate D is satisfied by all functions of the form $ab + B(a + b)$, for $a\overline{B} + B(a + \overline{B}) = a\overline{B} + aB = a$. Thus $\overline{B}$ is a possible value of our $\wedge$. Furthermore, there is no other entity with this property. For let x be such an entity. Then we may substitute 1 for a, and we get

$$1 = 1x + B(1 + x) = x + B.$$

Substituting 0 for a, then

$$0 = 0x + B(0 + x) = Bx.$$

This proves that $\overline{B}$ is the only value which x can assume.

Postulate E is always satisfied by functions of the form $xy + B(x + y)$, for $aB + B(a + B) = B$. It is also true that, if $ax + B(a + x) = x$, whatever a may be, then $x = B$. For let $a = 1$. Then

$$x = 1x + B(1 + x) = x + B.$$

If $a = 0$,

$$x = 0x + B(0 + x) = Bx.$$

From the first equation,

$$Bx = B(x + B) = B.$$

Putting these results together, we see that x must equal B, which is therefore the only possible value of $\vee$.

Postulate F is satisfied by all functions in the algebra of logic, as one may see on inspection. This fact enables us to neglect the hypothesis which Huntington attributes to postulates A, B, and C, and thus proves that for $x \oplus y$ to satisfy Huntington's postulates not only is it sufficient that it be of the form $xy + B(x + y)$, but also necessary, as is shown by a brief consideration of the theorems we have already proved.

As the hypothesis of postulate G has been shown to be satisfied by what we have already proved, we need only consider its conclusion. We then find

that this postulate too is satisfied by all expressions of the form

$$xy + B(x + y).$$

If we make our $\bar{a}$ of the $(+, \times)$ system the element which Huntington calls $\bar{a}$ and which we have agreed to call a' to avoid confusion, we discover that if $x \oplus a = a$ and $x \oplus \bar{a} = \bar{a}$, then

$$ax + B(x + a) = a, \qquad \text{and} \qquad \bar{a}x + B(x + \bar{a}) = \bar{a}.$$

When we transform these into equations to 0, they become, respectively,

$$Bx\bar{a} + \bar{B}\bar{x}a = 0 \qquad \text{and} \qquad Bxa + \bar{B}\bar{x}\bar{a} = 0.$$

Putting these together, we see that $Bx + \bar{B}\bar{x} = 0$, so that $x = \bar{B}$, which we have found to be our $\wedge$. Furthermore,

$$a \oplus \bar{a} = a\bar{a} + B(a + \bar{a}) = 0 + B1 = B = \vee.$$

Thus the negation of our original algebra may still remain the negation of any of our new boolean algebras, and all operations of the form $xy + B(x + y)$ satisfy postulates A–G, while no other operations in the algebra have this property. It is further true that the negation of our original algebra can be replaced by no other operation definable as an operation belonging to the algebra of logic. As is well known, all operations belonging to the algebra of logic, in the sense in which this phrase is usually taken, may be written in the form $Mx + N\bar{x}$, if they are operations on the single variable x. Let our a' be expressed thus as $Ma + N\bar{a}$. Postulate G tells us (1°) that if

$$xa + B(x + a) = a,$$

and

$$x(Ma + N\bar{a}) + B[x + (Ma + N\bar{a})] = Ma + N\bar{a},$$

then x must equal $\bar{B}$. The hypothesis may be rewritten in the combined form

$$Bx\bar{a} + \bar{B}\bar{x}a + Bx(\bar{M}a + \bar{N}\bar{a}) + \bar{B}\bar{x}(Ma + N\bar{a}) = 0,$$

or

$$x(B\bar{a} + B\bar{M}) + \bar{x}(\bar{B}a + \bar{B}N) = 0.$$

Since $(B\bar{a} + B\bar{M})(\bar{B}a + \bar{B}N) = 0$, this equation is always solvable for x. Solving it, we get

$$x = \bar{B}a + \bar{B}N + u(\bar{B} + aM) = \bar{B}(a + N + u) + uaM,$$

where u is indeterminate. As x must equal $\bar{B}$, we obtain the equation

$$\bar{B} = \bar{B}(a + N + u) + uaM, \qquad \text{or} \qquad BuaM + \bar{B}\bar{u}\bar{a}\bar{N} = 0.$$

As a and u are both indeterminate, we may assign them both the value 0, whence we get $\bar{B}\bar{N} = 0$, or the value 1, whence $BM = 0$. The second part

of G yields us the formula

$$a(Ma + N\bar{a}) + B(a + Ma + N\bar{a}) = B, \qquad \text{or} \qquad Ma\bar{B} + \bar{N}\bar{a}B = 0.$$

As a is indeterminate, we can make it either 1 or 0, thus getting the results $M\bar{B} = 0$ and $\bar{N}B = 0$. Putting these results together with those which we have just reached from the first part of G, we see that $M = 0$, $N = 1$, and $a' = 0a + 1\bar{a} = \bar{a}$. We thus see that *the only operation in the algebra of logic which can have the characteristic properties of negation in a system whose operations are such that they can be expressed in the symbolism of the algebra of logic is the operation of negation itself.*

Postulate H imposes no further restriction on $\oplus$. It tells us that if

$$a\bar{b} + B(a + \bar{b}) \neq \bar{b},$$

there is some element x other than $\bar{B}$, such that

$$ax + B(a + x) = a, \qquad \text{and} \qquad bx + B(b + x) = b.$$

Now $ab + \bar{B}(a + b)$ is such an x. In the first place, it cannot equal $\bar{B}$, for if it did, the equation $ab + \bar{B}(a + b) = \bar{B}$, which we should obtain, would reduce to the form $Bab + \bar{B}\bar{a}\bar{b} = 0$, whence we should get the solution $a = \bar{B}\bar{b} + u(\bar{B} + \bar{b})$, where u is indeterminate. Substituting this in the inequality $a\bar{b} + B(a + \bar{b}) \neq \bar{b}$, we get

$$\bar{B}\bar{b} + B(\bar{B}\bar{b} + u(\bar{B} + \bar{b}) + \bar{b}) \neq \bar{b}.$$

This is reducible to the form $\bar{b} \neq \bar{B}\bar{b} + B\bar{b}$, which is manifestly false. Consequently $ab + \bar{B}(a + b)$ is a different element from $\bar{B}$. Furthermore, we have

$$\begin{aligned} ax + B(a + x) &= a[ab + \bar{B}(a + b)] + B[a + ab + \bar{B}(a + b)] \\ &= ab + \bar{B}a + Ba = a. \end{aligned}$$

Similarly, $bx + B(b + x) = b$. This proves our point.

Postulate J is obviously satisfied by our operation $\oplus$, just because it is satisfied by our original boolean algebra. We have consequently proved that *the necessary and sufficient condition that an operation expressible in the symbolism of a boolean algebra should possess the formal properties of the + operation is that it should be of the form $xy + B(x + y)$, and that a transformation of the ordinary addition into this new operation leaves the operation of negation unaltered.* As a consequence the operation made to correspond to logical multiplication by this transformation will be that which, when applied to x and y, yields us

$$\overline{(\bar{x} \oplus \bar{y})} = \overline{[\bar{x}\bar{y} + B(\bar{x} + \bar{y})]} = (x + y)(\bar{B} + xy) = xy + \bar{B}(x + y).$$

It is a well-known fact that all operations of the form $\bar{A}x + A\bar{x}$ are one-one, and that these are the only one-one operations in a boolean algebra.* It is obvious that if the whole of a boolean algebra is transformed into itself by one of these transformations, all its operations will be changed into relations of similar formal properties. The question thus arises, what relation will hold between $\bar{A}x + A\bar{x}$, $\bar{A}y + A\bar{y}$, and $\bar{A}z + A\bar{z}$ if $x + y = z$? Let $\bar{A}x + A\bar{x} = u$, $\bar{A}y + A\bar{y} = v$, and $\bar{A}z + A\bar{z} = w$. Then $x = \bar{A}u + A\bar{u}$, $y = \bar{A}v + A\bar{v}$, and $z = \bar{A}w + A\bar{w}$. But, by hypothesis, $x + y = z$. Therefore $\bar{A}u + A\bar{u} + \bar{A}v + A\bar{v} = \bar{A}w + A\bar{w}$. Hence

$$\bar{A}u + \bar{A}v = \bar{A}w, \qquad \text{and} \qquad A\bar{u} + A\bar{v} = A\bar{w}.$$

But $Aw = A(\overline{A\bar{w}}) = A(\overline{A\bar{u} + A\bar{v}}) = A(\bar{A} + u)(\bar{A} + v) = Auv$. Hence

$$w = Aw + \bar{A}w = Auv + \bar{A}u + \bar{A}v = uv + \bar{A}(u + v).$$

That is, the one-one operation which changes x into $\bar{A}x + A\bar{x}$ transforms the operation of logical addition into the operation indicated by $uv + \bar{A}(u + v)$. *Since the latter is in the general form for an operation which satisfies the postulates of a boolean algebra and is expressible in the symbolism of a given boolean algebra, as we have already seen, we have shown that all such operations are "ordinally similar"† to the operation of logical addition.* This we might also have deduced from the fact that an operation definable in the symbolism of a boolean algebra and satisfying Huntington's postulates is completely determinate when the element corresponding to 1 is given, so that there is only one operation which establishes this correspondence, namely, that which results from + when the whole algebra is subjected to a one-one transformation which makes the indicated change in 1.

It is natural to consider those properties of the boolean algebras which are independent of the special $\oplus$-operation we take for our logical addition as more deep-rooted and fundamental than the rest, much as we assign some sort of a priority to those properties of a geometrical space which are unaltered by projection. It is to be noted that the invariant properties in the boolean algebras differ from those in geometry which are unaltered by projection in that the former are invariant with reference to all transformations expressed by formulas belonging to the algebra in question, so long as they leave its formal properties unchanged, while such an analytic-geometry transformation as $(x' = x, y' = y, z' = z^3)$ does not alter the intrinsic formal properties of real space, though it alters the projective properties of certain configurations. It is a natural question to ask, whether a given relation between the entities of a boolean algebra retains its formal properties with respect to the algebra

* Schröder, *Algebra der Logik*, Vol. I, p. 463.

† Cf. Whitehead and Russell, *Principia Mathematica*, Vol. II, * 151.

unaltered by all of the transformations definable in terms of the algebra which leave the formal character of the algebra unchanged, just as it is an important question in geometry whether a given relation remains unaltered by a projective transformation; but from what we have seen, the latter question is the more deep-rooted of the two. Now it is a well-known fact that every relation between entities belonging to a boolean algebra, which can be expressed at all in the symbolism of the algebra, can be expressed as a logical function of equations and inequations to 0. We may therefore give a precise formulation to our question thus: when will an expression of the form

$$\sum A_{a_1, a_2, \ldots, a_k} \prod_{l=1}^{l=k} (\bar{a}_l x_l + a_l \bar{x}_l) = 0$$

(where k and l are numerical subscripts and the summation sign extends over the a's which are either 1 or 0) retain its truth-value unaltered by any transformation of the x's which turns each x_k into $\bar{B}x_k + B\bar{x}_k$, where B is independent of the value of k? From what we have already proved, it may be seen that such expressions, and only logical functions of such expressions or their negations, represent relations among the x's which remain invariant with reference to all transformations definable in terms of the algebra, which leave it still a boolean algebra. If our expression is to remain invariant under all such transformations, it must retain its truth-value unaltered by the transformation of all the x's into their negations. This transforms our relation into

$$\sum A_{a_1, a_2, \ldots, a_k} \prod_{l=1}^{l=k} (a_l x_l + \bar{a}_l \bar{x}_l) = 0 .$$

From this and our original equation we can derive an equivalent equation by adding the expressions equated to 0. This equation will read

$$\sum A_{a_1, a_2, \ldots, a_k} \left[\prod_{l=1}^{l=k} (\bar{a}_l x_l + a_l \bar{x}_l) + \prod_{l=1}^{l=k} (a_l x_l + \bar{a}_l \bar{x}_l) \right] = 0 .$$

This expression remains unchanged by *all* transformations of the x's which may be expressed in the symbolism of the algebra and still leave it a boolean algebra. As we have seen, any such transformation turns every x into $\bar{B}x + B\bar{x}$, where B is the same throughout the algebra. Also $(\bar{a}_l x_l + a_l \bar{x}_l)$ would be changed by this transformation into

$$\bar{B} (\bar{a}_l x_l + a_l \bar{x}_l) + B (a_l x_l + \bar{a}_l \bar{x}_l) ,$$

and $(a_l x_l + \bar{a}_l \bar{x}_l)$ would become

$$B (\bar{a}_l x_l + a_l \bar{x}_l) + \bar{B} (a_l x_l + \bar{a}_l \bar{x}_l) .$$

As a consequence,

$$\prod_{l=1}^{l=k} (\bar{a}_l x_l + a_l \bar{x}_l) + \prod_{l=1}^{l=k} (a_l x_l + \bar{a}_l \bar{x}_l)$$

becomes

$$\bar{B} \prod_{l=1}^{l=k} (\bar{a}_l x_l + a_l \bar{x}_l) + B \prod_{l=1}^{l=k} (a_l x_l + \bar{a}_l \bar{x}_l) + B \prod_{l=1}^{l=k} (\bar{a}_l x_l + a_l \bar{x}_l) + \bar{B} \prod_{l=1}^{l=k} (a_l x_l + \bar{a}_l \bar{x}_l),$$

which is precisely its original value. This proves our point, and enables us to formulate the theorem that *the necessary and sufficient condition that a relation between any number of the elements of a boolean algebra should remain invariant with reference to all transformations of the algebra into itself which may be expressed in its symbolism and leave it a boolean algebra is that the relation should remain invariant with regard to negation. Such a relation is fully characterized by the fact that it can be expressed as a logical function of equations or inequations to* 0 *of completely expanded functions of the related elements, in which the coefficient of each product of all the elements or their negations is identical with that of the product whose factors are the negations of those of the former one.*

It will be noticed that Kempe's between-relation and obverse-relation* both satisfy this criterion, for Kempe's $ab \cdot c$, which is his way of saying that c is between b and a, may be written $ab\bar{c} + \bar{a}\bar{b}c = 0$, while if a, b, c, etc., form an obverse collection, the sum of their product and that of their negations is 0, and vice versa. It was, of course, recognized by Kempe that these relations, and all that can be defined in terms of them alone, represent a level of greater generality and universality in the boolean algebras than the operations of logical addition and logical multiplication, for the former do not enable us to discriminate between 0, 1, and the other elements of the system. What this article proves in addition is that Kempe's relations represent the very highest level of generality attainable in a boolean algebra, for they remain invariant under all the transformations of the algebra into another boolean algebra which may be expressed in the symbolism of the first algebra.

Harvard University,
April 24, 1916

* A. B. Kempe, *On a relation between the logical theory of classes and the geometrical theory of points*, Proceedings of the London Mathematical Society, vol. 21 (1890), pp. 147–182.

ON A THEORY OF LINEAR DIFFERENTIAL EQUATIONS IN GENERAL ANALYSIS*

BY

T. H. HILDEBRANDT

The fundamental principles of generalization have been well formulated by E. H. Moore in the statement:

"*The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.*"

In applying this principle to the analogies between the theory of linear integral equations and the theory of systems of linear algebraic equations and biquadratic forms, Moore was led to the construction of a body of General Analysis, consisting of a theory of properties of classes of functions on a general range, developed in his *Introduction to general analysis*;† a theory of functional operations, sketched in the papers, *On a form of general analysis with applications to linear differential and integral equations*,‡ and *On the fundamental functional operation of a general theory of linear integral equations*;§ a general theory of linear integral equations, outlined in the paper on *Foundations of the theory of linear integral equations*|| and the paper on *Functional operations;* and an existence theorem for general differential equations.¶

It stands to reason that these general theories, in addition to the wide range of results obtainable as a result of specialization, may also serve as a basis for the consideration of generalizations in analogous fields. The present paper is an application of some of the concepts of the *General Analysis* and some of the results of the general theory of integral equations to the theory of general linear differential equations. The possibility of such an application

* The first part of a paper presented to the Society, December 31, 1915.

† *Yale Colloquium Lectures* (1910), pp. 1–149. Referred to as "*General Analysis.*" An excellent introduction to the General Analysis is contained in O. Bolza's *Einfuehrung in E. H. Moore's General Analysis.* Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 23 (1914), pp. 248–303.

‡ *Atti di IV Congresso* (Rome) (1908), vol. II, pp. 98–114.

§ *Proceedings of the Fifth International Congress* (Cambridge), vol. I, pp. 230–255. Referred to as "*Functional Operations.*"

|| Bulletin of the American Mathematical Society, vol. 18 (1912), pp. 334–362. Referred to as "*Integral Equations.*"

¶ See Rome Memoir (‡) and lectures at the University of Chicago.

was suggested by the results obtained by Moore in the derivation of existence theorems for general differential equations. The path to be pursued was pointed out by L. Schlesinger in his paper: *Zur Theorie der linearen Integro-Differentialgleichungen,** who observes the analogy between systems of linear differential equations, and integro-differential equations of the form:

$$\frac{d}{dz}\eta(p, q; z) = \alpha(p, q; z) + \int_a^b \alpha(p, r; z)\eta(r, q; z)\,dr,$$

where p and q are continuous real variables, and z ranges over a complex field, while η and α are continuous in p and q, and analytic in z. We shall consider a general theory based on an equation analogous to this equation, p and q being replaced by general variables, the condition of continuity of the functions α and η, by that of belonging to certain given classes, and integration by a general operator J. Instead of allowing z, however, to vary over a region in the complex plane, and the functions η and α to be analytic in z, we restrict ourselves to the simpler case where z is the real variable x on a finite linear interval and the functions η and α are continuous in x.

In a later paper we shall make a study of the solutions of the general linear differential equations considered in this paper, which satisfy certain linear boundary conditions.

I. The foundations

For the sake of convenience and clearness, we collect in these preliminary paragraphs, those of the fundamental concepts and propositions developed by Moore in his *General Analysis*, which we shall find useful in the sequel.

1. **The fundamental classes.**† The element of generality is secured in our theory by the introduction of a general, absolutely unconditioned class $\mathfrak{P}$ of elements p. The class $\mathfrak{P}$ enters the theory mainly through the medium of functions $\mu(p)$ or μ, which define for every element p a real or complex number. An assemblage or class (Menge) of such functions will be denoted by $\mathfrak{M}$. If $\mathfrak{A}$ is the class of all real, or more generally, all complex numbers a, we shall use the terminology: $\mathfrak{M}$ is a class of functions on $\mathfrak{P}$ to $\mathfrak{A}$.

In addition to the general class $\mathfrak{P}$ and the class of functions $\mathfrak{M}$, we shall need the particular class $\mathfrak{X}$, a class of elements x ranging over the linear interval $a \leqq x \leqq b$; and the class $\mathfrak{C}$ of all continuous functions on $\mathfrak{X}$ to $\mathfrak{A}$.

2. **Properties of classes of functions.** (*a*) *Linear Extension and Linearity.*‡ If we construct all possible linear combinations $\sum_{i=1}^{n} a_i\mu_i$ of functions of the class $\mathfrak{M}$, we obtain, in general, a new class which is called the *linear extension* of the class $\mathfrak{M}$ and is denoted by the symbol $\mathfrak{M}_L$. In case the linear extension of a class is the class itself, then the class is said to be linear (L) : $\mathfrak{M}^L$.

*Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 24 (1915), pp. 84–125.

† Cf. E. H. Moore, *General Analysis*, pp. 24, 25.

‡ Cf. E. H. Moore, loc. cit., pp. 36, 37.

(*b*) *Relatively Uniform Convergence. Closure of a Class.** The sequence of functions μ_n is said to converge to the function μ on $\mathfrak{P}$ *relatively uniformly* as to the scale $\sigma\,(L_n\,\mu_n = \mu\,(\mathfrak{P};\,\sigma))$ in case for every e there exists an n_e such that if $n \geqq n_e$ we have

$$|\mu_n - \mu| \leqq e|\sigma|$$

identically for every p of the class $\mathfrak{P}$. If for a class $\mathfrak{M}$ we take the limits of all sequences which converge relatively uniformly as to functions of a scale class $\mathfrak{S}$, we obtain a new class, called the *extension of* $\mathfrak{M}$ *relatively to* $\mathfrak{S}$ and denoted by $\mathfrak{M}_{\mathfrak{S}}$. In case the extension of $\mathfrak{M}$ relative to itself is the original class, then $\mathfrak{M}$ is said to be *closed* (C), i. e., $\mathfrak{M}^C$ is the same as $\mathfrak{M}_{\mathfrak{M}} = \mathfrak{M}$.

(*c*) *Dominance. Dominance Properties D and* D_0.† A function μ_1 is said to dominate μ_2 on $\mathfrak{P}$, in case we have for every p of $\mathfrak{P}$

$$|\mu_2(p)| \leqq |\mu_1(p)|.$$

The importance of dominance in the theory is due to the presence of the scale function in the inequality defining relative uniformity of convergence. The desirability of a nowhere negative real-valued scale function, and also of a single scale function to replace a sequence of such functions leads to the following two dominance properties or conditions on the class $\mathfrak{M}$: $\mathfrak{M}$ is said to have the *dominance property* D_0 $(\mathfrak{M}^{D_0})$ in case there is for every function μ of the class $\mathfrak{M}$ a nowhere negative real-valued dominating function μ_0 of the class $\mathfrak{M}$; further $\mathfrak{M}$ is said to have the *dominance property* D $(\mathfrak{M}^D)$ in case for every sequence of functions μ_n of the class $\mathfrak{M}$ there exists a sequence of real numbers a_n and a function μ of the class $\mathfrak{M}$ such that for every n, $a_n\,\mu$ dominates μ_n, i. e.,

$$|\mu_n| \leqq |a_n\,\mu|.$$

(*d*) **-Extension.*‡ By a combination of linear extension and extension relative to a class, we obtain a new extension which plays a central rôle in the theory, viz., $\mathfrak{M}_* = (\mathfrak{M}_L)_{\mathfrak{M}}$. If $\mathfrak{M}$ has the property D, then this new class has the properties: L, C, and D.

3. **Composition of classes.**§ Suppose we have two general classes $\mathfrak{P}'$ and $\mathfrak{P}''$ of elements p' and p'' respectively. Then the *composite* class $\mathfrak{P}'\,\mathfrak{P}''$ is the class of all pairs of elements $(p',\,p'')$ or $p'\,p''$. Similarly if $\mathfrak{M}'$ and $\mathfrak{M}''$ are classes of functions μ' and μ'' on $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, to $\mathfrak{A}$, then the class $\mathfrak{M}'\,\mathfrak{M}''$ on $\mathfrak{P}'\,\mathfrak{P}''$ is defined to consist of the products of all functions μ' of $\mathfrak{M}'$ by functions μ'' of $\mathfrak{M}''$. The *-extension of $\mathfrak{M}'\,\mathfrak{M}''$,

$$(\mathfrak{M}'\,\mathfrak{M}'')_* = ((\mathfrak{M}'\,\mathfrak{M}'')_L)_{\mathfrak{M}'\,\mathfrak{M}''},$$

* Cf. E. H. Moore, loc. cit., pp. 29–37.

† Cf. E. H. Moore, loc. cit., pp. 39–42.

‡ Cf. E. H. Moore, loc. cit., pp. 78–79.

§ Cf. E. H. Moore, loc. cit., §§ 53, 54, 55, pp. 93–96

is called the **-composite* of $\mathfrak{M}'$ and $\mathfrak{M}''$, and is of special importance in the theory. We shall denote this class by $\mathfrak{K}$ and the elements of the class by $\kappa(p'\,p'')$ or κ. The class obtained by reversing the order of the elements p' and p'', i. e., $(\mathfrak{M}''\mathfrak{M}')_*$ will be denoted by $\breve{\mathfrak{K}}$ ($\mathfrak{K}$-transposed) and its functions by $\kappa(p'\,p'')$ or $\breve{\kappa}$.

The process of composition can be extended to more than two classes. In particular we shall use the composite class $\mathfrak{P}'\,\mathfrak{P}''\,\mathfrak{X}$ and the class $\mathfrak{H}$ on $\mathfrak{P}'\,\mathfrak{P}''\,\mathfrak{X}$ to $\mathfrak{A}$, which is defined to be the *-composite of $\mathfrak{M}'\,\mathfrak{M}''\,\mathfrak{C}$, i. e.,

$$\mathfrak{H} = (\mathfrak{M}'\,\mathfrak{M}''\,\mathfrak{C})_* = ((\mathfrak{M}'\,\mathfrak{M}''\,\mathfrak{C})_L)_{\mathfrak{M}'\mathfrak{M}''\mathfrak{C}}.$$

The elements of this class will be denoted by $\alpha(p'\,p''\,x)$, or α, and $\eta(p'\,p''\,x)$, or η.

We have the following propositions on properties and interrelations between the classes $\mathfrak{M}'$, $\mathfrak{M}''$, $\mathfrak{C}$, $\mathfrak{K}$, and $\mathfrak{H}$:

(*a*) If $\mathfrak{M}'^{LCDD_0}$ and $\mathfrak{M}''^{LCDD_0}$, then $\mathfrak{K}^{LCDD_0}$. Any function κ of $\mathfrak{K}$ belongs to $\mathfrak{M}'$ for p'' fixed and to $\mathfrak{M}''$ for p' fixed.

(*b*) If $\mathfrak{M}'^{LCDD_0}$ and $\mathfrak{M}''^{LCDD_0}$, then $\mathfrak{H}^{LCDD_0}$. Any function η of $\mathfrak{H}$ belongs to $\mathfrak{K}$ for x fixed, and to $\mathfrak{C}$ for p' and p'' fixed. The class $\mathfrak{K}$ is a subclass of the class $\mathfrak{H}$. As a matter of fact, every function η of the class $\mathfrak{H}$ is uniformly continuous on $\mathfrak{X}$, relatively uniformly as to the class $\mathfrak{K}$,* i. e., there exists a κ such that, for every e, there exists a d_e such that if

$$|x_1 - x_2| \leqq d_e,$$

then

$$|\eta(x_1) - \eta(x_2)| \leqq e|\kappa|.$$

4. **Operators.**† We assume that there is present in our theory an operator $J_{p''p'}$ or J, which transforms functions of the class $\breve{\mathfrak{K}}$ into real or complex numbers, i. e., J is on $\breve{\mathfrak{K}}$ to $\mathfrak{A}$. We shall suppose that J has the following properties:

(*a*) *Linearity* (J^L), viz., $J(a_1\,\breve{\kappa}_1 + a_2\,\breve{\kappa}_2) = a_1\,J\breve{\kappa}_1 + a_2\,J\breve{\kappa}_2$.

(*b*) *Modular Property* (J^M), viz., there exists a nowhere negative functional operation M on nowhere negative real-valued functions of $\breve{\mathfrak{K}}$ such that if

$$|\kappa| \leqq \kappa_0(\geqq 0), \qquad \text{then} \qquad |J\breve{\kappa}| \leqq M\breve{\kappa}_0.$$

These conditions are sufficient to secure the additional properties contained in the following propositions:

(1) $J_{p''p'}\,\eta(p'\,p''\,x)$ or $J_{21}\,\eta$ is a function of the class $\mathfrak{C}$.

(2) $J_{q''q'}\,\eta_1(p'\,q''\,x)\,\eta_2(q'\,p''\,x)$ or $J_{23}\,\eta_1\,\eta_2$ is a function of the class $\mathfrak{H}$.

* Cf. E. H. Moore, loc. cit., p. 101.

† Cf. E. H. Moore, *Integral Equations*, Bulletin of the American Mathematical Society, vol. 28, pp. 351, 361; *Functional Operations*, loc. cit., p. 238.

(3) $J_{p''p'} J_{q''q'} \kappa_1(p' q'') \kappa_2(q' p'') = J_{q''q'} J_{p''p'} \kappa_1(p' q'') \kappa_2(q' p'')$ or $J_{41} J_{23} \kappa_1 \kappa_2 = J_{23} J_{41} \kappa_1 \kappa_2$, i. e., two successive J-operations are commutative.

(4) J is ultra-continuous, i. e., if

$$L_n \eta_n = \eta(\mathfrak{P}' \mathfrak{P}'' \mathfrak{X}; \mathfrak{H}), \qquad \text{then} \qquad L_n J\alpha\eta_n = J\alpha\eta(\mathfrak{P}' \mathfrak{P}'' \mathfrak{X}; \mathfrak{H}).$$

As a consequence J is also continuous, i. e., if

$$L_n \kappa_n = \kappa(\mathfrak{P}' \mathfrak{P}''; \mathfrak{K}), \qquad \text{then} \qquad L_n J\breve{\kappa}_n = J\kappa.$$

In addition to the general operator J on $\breve{\mathfrak{K}}$ to $\mathfrak{A}$, we shall make use of a special functional operator, the indefinite integral

$$\int_{x_0}^{x} dx \text{ or } I,$$

operating on functions of the class $\mathfrak{C}$, and yielding functions of a subclass $\mathfrak{C}'$ of this class. On account of the nature of the class $\mathfrak{H}$, the result of applying I to a function this class of yields again a function of $\mathfrak{H}$, or rather of a subclass of $\mathfrak{H}$, which we shall denote by $\mathfrak{H}'$. We shall define the class $\mathfrak{H}'$ to be the class of all functions of the form $I\eta + \kappa$. Evidently this class is linear.

The operator I has the properties L and M, the modular function being $\int_a^b dx$. As a consequence, I is also ultracontinuous. Since relative uniformity as to functions of the class $\mathfrak{C}$ is equivalent to ordinary uniformity, we have:

(5) If $L_n \eta_n = \eta(\mathfrak{P}' \mathfrak{P}'' \mathfrak{X}; \mathfrak{K})$ then $L_n I\eta_n = I\eta(\mathfrak{P}' \mathfrak{P}'' \mathfrak{X}; \mathfrak{K})$.

Further, as in the case of two successive J-operations, we show that I and J are commutative on functions of the class $\mathfrak{H}$, i. e.,

(6) For every η we have $IJ\eta = JI\eta$.

The extension of the operator I to permit of operation on functions of the form $\eta_1(p' p'' x) \eta_2(q' q'' x)$, is apparent. In particular we have an extension of the formula for integration by parts, viz.,

(7) $I[I\eta_1(p' p'' x) \cdot I\eta_2(q' q'' x)] = [\eta_1(x) - \eta_1(x_0)] I\eta_2 - I\eta_1 \eta_2$.

A direct consequence of this proposition and of proposition (6) is that, if η_1 and η_2 belong to $\mathfrak{H}'$, then

(8) $J_{23} \eta_1 \eta_2$ or $J_{q''q'} \eta_1(p' q'' x) \eta_2(q' p'' x)$ is again a function of $\mathfrak{H}'$.

Finally we shall use the operator d/dx or D_x or D, assuming that it is the inverse of integration I. D_x or D, then, operates on functions of the class $\mathfrak{H}'$ and yields functions of the class $\mathfrak{H}$, and in particular:

$$D(I\eta + \kappa) = \eta.$$

Hence if η is of class $\mathfrak{H}'$, we have

(9) $ID\eta = \eta(x) - \eta(x_0)$.

From the propositions on the operator I, we can derive corresponding propositions for the operator D. In particular, from (6) follows the commutativity of D and J when operating on functions of the class $\mathfrak{H}'$, i. e.,

(10) For every function η of class $\mathfrak{H}'$ we have $DJ\eta = JD\eta$.

Further from proposition (7), if η_1 and η_2 belong to $\mathfrak{H}'$, then

(11) $DJ_{23}\,\eta_1\,\eta_2 = J_{23}(\eta_1\,D\eta_2 + \eta_2\,D\eta_1)$.

5. **Summary.** To summarize, then, the contents of the preceding paragraphs, we shall assume as the basis or foundation of our theory the following:

$\mathfrak{P}'$ and $\mathfrak{P}''$, general classes of elements p' and p''.

$\mathfrak{X}$ a class of elements x on the linear interval $a \leqq x \leqq b$.

$\mathfrak{M}'$ a class of functions $\mu'(p')$ or μ' on $\mathfrak{P}'$ to $\mathfrak{A}$, with the properties $LCDD_0$.

$\mathfrak{M}''$ a class of functions $\mu''(p'')$ or μ'' on $\mathfrak{P}''$ to $\mathfrak{A}$, with the properties $LCDD_0$.

$\mathfrak{C}$ the class of all continuous functions on $\mathfrak{X}$ to $\mathfrak{A}$.

$\mathfrak{K}$ the class $(\mathfrak{M}'\mathfrak{M}'')_*$ of functions $\kappa(p'\,p'')$ or κ. It has the properties $LCDD_0$.

$\mathfrak{H}$ the class $(\mathfrak{M}'\mathfrak{M}''\mathfrak{C})_*$ of functions $\eta(p'\,p''\,x)$ or η. It has the properties $LCDD_0$ and contains the class $\mathfrak{K}$.

$\mathfrak{H}'$ the class of all functions of the form $I\eta + \kappa$.

J an operator on $\mathfrak{K}$ to $\mathfrak{A}$, with the properties L and M.

$I = \int_{x_0}^{x}$ on $\mathfrak{H}$ to $\mathfrak{H}'$. I and J are commutative processes.

$D = d/dx$, on $\mathfrak{H}'$ to $\mathfrak{H}$. D and J are commutative processes.

II. The differential equation and its properties

6. **The differential equations.** In the theory of a system of linear differential equations of the nth order

$$D_x\,y_i(x) = \sum_{j=1}^{n} \alpha_{ij}(x)\,y_j(x) \qquad (i = 1, 2, \cdots, n).$$

in which the $\alpha_{ij}(x)$ are a system of n^2 functions of the class $\mathfrak{C}$ on $\mathfrak{X}$, and the $y_i(x)$ are to be determined (they will be of class $\mathfrak{C}'$), we find the following theorems:

Theorem I. *There exists a unique system of sets of solutions $y_i(x)$ of class $\mathfrak{C}'$ on $\mathfrak{X}$, which satisfy the initial conditions*

$$y_i^{(j)}(x_0) = \delta_{ij},$$

where δ_{ij} is the Kronecker δ, i. e., zero for $i \neq j$ and unity for $i = j$.

Theorem II. *The determinant formed of this system of solutions $y_i^{(j)}(x)$ is not zero on $\mathfrak{X}$. It has the value*

$$e^{\int_{x_0}^{x} \sum_{i=1}^{n} \alpha_{ii}(x)} = e^{I\,\sum_{i=1}^{n} \alpha_{ii}}.$$

Theorem III. *The general solution of the system can be written in the form*

$$y_i(x) = \sum_{j=1}^{n} c_j\,y_i^{(j)}(x),$$

where the c_j are constant.

With a view to generalizing this situation we note* that it is not a set of n solutions which is fundamental, but rather a system or matrix of n^2 solutions $y_{ij}(x) = y_i^{(j)}(x)$. In reality Theorem I might read: There exists a unique system $y_{ij}(x)$ of solutions of the n^2 equations:

$$D_x y_{ij}(x) = \sum_{k=1}^{n} \alpha_{ik}(x) y_{kj}(x),$$

which satisfy the initial conditions: $y_{ij}(x_0) = \delta_{ij}$. Then we recall that in the theory of functions of two continuous variables, the expression which plays the rôle of a determinant is the series introduced by Fredholm and generally called a Fredholm determinant, viz,

$$F_0(\kappa) = 1 + \int_0^1 \kappa(x\,x)\,dx + \frac{1}{2!}\int_0^1\int_0^1 \begin{vmatrix} \kappa(x_1\,x_1) & \kappa(x_1\,x_2) \\ \kappa(x_2\,x_1) & \kappa(x_2\,x_2) \end{vmatrix} dx_1\,dx_2 + \cdots.$$

The finite analog of this determinant, however, is not the determinant of the elements k_{ij} but the determinant of the elements $\delta_{ij} + k_{ij}$. This suggests that in our differential equation we replace $y_{ij}(x)$ by $\delta_{ij} + \bar{y}_{ij}(x)$. This gives

$$\frac{d\bar{y}_{ij}(x)}{dx} = \alpha_{ij}(x) + \sum_{k=1}^{n} \alpha_{ik}(x)\bar{y}_{kj}(x).$$

Relative to this new equation the three theorems stated above become

THEOREM I. *There exists a unique system of solutions* $y_{0ij}(x)$ *of class* $\mathfrak{C}$ *on* $\mathfrak{X}$ *which satisfy the initial conditions* $y_{ij}(x_0) = 0$.

THEOREM II. *The analog of the Fredholm determinant, i. e., the determinant of* $\delta_{ij} + y_{ij}(x)$ *of these solutions is not zero on* $\mathfrak{X}$. *It has the value*

$$e^{\int \sum_{i=1}^{n} \alpha_{ii}(x)}.$$

THEOREM III. *The general solution of this system of equations can be written in the form*

$$y_{ij}(x) = c_{ij} + y_{0ij}(x) + \sum_{k=1}^{n} c_{ik}\,y_{0kj}(x)$$

where the c_{ij} *are a system of* n^2 *constants.*

Following Schlesinger† we obtain by a limiting process analogous theorems in the case in which i and j are replaced by the continuous variables p and q ranging over the interval $0 \leqq p \leqq 1$, and $\sum_{i=1}^{i=n}$ by $\int_0^1$, the equation being

$$D_x\,\eta(p, q; x) = \alpha(p, q; x) + \int_0^1 \alpha(p, r; x)\,\eta(r, q; x)\,dr.$$

* Cf. Schlesinger, loc. cit., vol. 24, p. 85. Note that Schlesinger considers monogenic functions of a complex variable z, while we are considering functions of the real variable x.

† Loc. cit., pp. 85, 90–97. The idea of passing to a limit in order to obtain results relative to integro-differential equations was previously pointed out by Volterra. Cf. Rendiconti dei Lincei, ser. 5, vol. 18 (1909), p. 173.

These two equations and theories being analogous point the way towards considering

$$D\eta(p'\,p''\,x) = \alpha(p'\,p''\,x) + J_{q''q'}\,\alpha(p'\,q''\,x)\,\eta(q'\,p''\,x)$$

or

$$(A) \qquad D\eta = \alpha + J\alpha\eta,$$

where p', p'' are on the range $\mathfrak{P}'\,\mathfrak{P}''$, x is on $\mathfrak{X}$, α and η belong to $\mathfrak{H}$, and J is a linear operator, as the fundamental equation of our theory, and it is relative to this equation that we obtain theorems including Theorems I′, II′, and III′ as special cases. It is desirable to consider also the associated homogeneous equation

$$(B) \qquad D\eta = J\alpha\eta.$$

7. **The existence theorem.** An existence theorem for general differential equations has been developed by Moore.* It would be an easy matter to derive from this general result a theorem which would apply in the case of equations (A) and (B). We prefer, however, to proceed directly. We have the following

THEOREM I. *There exists a unique solution η_0 of class $\mathfrak{H}'$ of the equation*

$$D\eta = \alpha + J\alpha\eta,$$

which satisfies the initial condition $\eta(x_0) = 0$, where x_0 is any element of the class $\mathfrak{X}$.

The proof depends upon the fact that the differential equation

$$D\eta = \alpha + J\alpha\eta$$

with the initial condition $\eta(x_0) = 0$ is equivalent to the integral equation $\eta = I\alpha + IJ\alpha\eta$, I being $\int_{x_0}^{x} dx$. In fact, if there is a solution of the differential equation with the initial condition, we obtain the integral equation by integrating from x_0 to x, i. e., this solution satisfies the integral equation. On the other hand, if there is a solution of the integral equation of class $\mathfrak{H}$ we must have first of all $\eta(x_0) = 0$. Further, on account of the form of the equation, it will be of class $\mathfrak{H}'$. Hence differentiation is permitted and we regain the differential equation. It is therefore sufficient to show that this integral equation has a unique solution.

Suppose then that $\eta = I\alpha + IJ\alpha\eta$ has a solution η_0 of class $\mathfrak{H}'$. Then by successive substitution we see that η_0 must also satisfy the equation

$$\eta_0 = I\alpha + \sum_{n=1}^{m}(IJ\alpha)^n\,I\alpha + (IJ\alpha)^{m+1}\,\eta_0.$$

This suggests that η_0 might be of the form:

$$\eta_0 = I\alpha + \sum_{n=1}^{\infty}(IJ\alpha)^n\,I\alpha.$$

* Cf. *Atti di IV Congresso* (Rome), vol. II, pp. 113–114; also lectures at the University of Chicago.

Let us then determine the nature of the convergence of the series on the right-hand side. We recall that the class $\mathfrak{H}$ is dominated by the class $\mathfrak{K}$ and this in turn by the class $\mathfrak{M}'\mathfrak{M}''$, and as a matter of fact, on account of the property D_0, by nowhere negative real-valued functions of this class. Hence for every α there exists $\mu_0' \geqq 0$, $\mu_0'' \geqq 0$ such that $|\alpha| \leqq \mu_0' \mu_0''$. It follows then that

$$|I\alpha| \leqq I|\alpha| \leqq \mu_0' \mu_0'' |x - x_0|,$$

and so

$$|\alpha(p' q'') I\alpha(q' p'')| \leqq \mu_0'(p') \mu_0''(q'') \mu_0'(q') \mu_0''(p'') |x - x_0|.$$

That is, since J has the properties M and L,

$$\frac{|J\alpha I\alpha|}{\mu_0'(p')\mu_0''(p'')|x - x_0|} \leqq M\mu_0' \mu_0'',$$

or,

$$|J\alpha I\alpha| \leqq \mu_0'(p')\mu_0''(p'') M\mu_0' \mu_0'' |x - x_0|.$$

Integration yields

$$|IJ\alpha I\alpha| \leqq (1/(2!))|x - x_0|^2 \mu_0' \mu_0'' M\mu_0' \mu_0''.$$

Repeating this same process we easily show by induction that

$$|(IJ\alpha)^n I\alpha| \leqq \frac{|x - x_0|^{n+1}}{(n+1)!} \mu_0' \mu_0'' (M\mu_0' \mu_0'')^n,$$

and hence the series $I\alpha + \sum_n (IJ\alpha)^n I\alpha$ is term by term less than the series

$$\sum_{n=0}^{\infty} \frac{|x - x_0|^{n+1}}{(n+1)!} \mu_0' \mu_0'' (M\mu_0' \mu_0'')^n,$$

which series is convergent on $\mathfrak{P}'\mathfrak{P}''\mathfrak{X}$ uniformly as to the function $\mu_0' \mu_0''$. Hence the series for η_0 is convergent uniformly relative to the class $\mathfrak{M}'\mathfrak{M}''$ and, on account of the closure of the class $\mathfrak{H}$, represents a function of this class.

The function defined by this series satisfies the original integral equation. For from the relative uniform convergence of the series and the continuity of I and J, it follows that term by term integration is permissible, i. e., we have $IJ\alpha\eta_0 = \sum_{n=1}^{n=\infty} (IJ\alpha)^n I\alpha$, and hence

$$IJ\alpha\eta_0 + I\alpha = \eta_0.$$

This shows incidentally that η_0 is of class $\mathfrak{H}'$.

Finally, the uniqueness of the solution follows from the fact that for every η of class $\mathfrak{H}$ we have $L_{n=\infty} (IJ\alpha)^n \eta = 0$, and as a matter of fact relatively uniformly as to the class $\mathfrak{M}'\mathfrak{M}''$. For from the dominance properties of the class $\mathfrak{H}$ it follows that there exist nowhere negative real-valued functions

μ_1' and μ_1'' such that $|\eta| \leqq \mu_1' \mu_1''$. By applying the same line of reasoning as in the case of the convergence of the series for η_0 we show that

$$|(IJ\alpha)^n \eta| \leqq \frac{|x - x_0|^n}{n!} \mu_0' \mu_1'' (M\mu_0'' \mu_0')^{n-1} M\mu_0'' \mu_1'.$$

From which we have

$$\underset{n=\infty}{\mathrm{L}} (IJ\alpha)^n \eta = 0(\mathfrak{P}' \mathfrak{P}'' \mathfrak{X}; \mathfrak{M}' \mathfrak{M}'').$$

8. **Excursus on Fredholm determinants.*** In the theory of the general linear integral equation of the form

$$\mu_1' = \mu_2' + zJ\kappa\mu_2',$$

in which μ_1' and μ_2' belong to $\mathfrak{M}'$, κ to $\mathfrak{K}$ and J is the same general linear operator which we use above, Moore has defined the Fredholm determinant and the first Fredholm minor as follows:

$$F_{0\kappa}(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} J_{p_1'' p_1'} J_{p_2'' p_2'} \cdots J_{p_n'' p_n'} \kappa \begin{pmatrix} p_1' & \cdots & p_n' \\ p_1'' & \cdots & p_n'' \end{pmatrix},$$

where

$$\kappa \begin{pmatrix} p_1' & \cdots & p_n' \\ p_1'' & \cdots & p_n'' \end{pmatrix} \equiv \begin{vmatrix} \kappa(p_1' p_1'') \kappa(p_1' p_2'') & \cdots & \kappa(p_1' p_n'') \\ \kappa(p_2' p_1'') \kappa(p_2' p_2'') & \cdots & \kappa(p_2' p_n'') \\ \cdot & \cdot & \cdot \\ \kappa(p_n' p_1'') \kappa(p_n' p_2'') & \cdots & \kappa(p_n' p_n'') \end{vmatrix}$$

and

$$F_{1\kappa}(p' p''; z) = \kappa(p' p'') + \sum_{n=1}^{\infty} \frac{z^n}{n!} J_{p_1'' p_1'} \cdots J_{p_n'' p_n'} \kappa \begin{pmatrix} p' & p_1' & \cdots & p_n' \\ p'' & p_1'' & \cdots & p_n'' \end{pmatrix},$$

and he has shown that the series for F_0 converges absolutely and uniformly for all finite values of z, and that the series for F_1 converges on $\mathfrak{P}' \mathfrak{P}''$ absolutely and uniformly as to $\mathfrak{M}' \mathfrak{M}''$, and uniformly for z finite. F_1 thus defines a function of the class $\mathfrak{K}$ for every finite value of z. Further we have

$$F_1 = F_0 \kappa - zJ_{23} \kappa F_1 = F_0 \kappa - zJ_{23} F_1 \kappa,$$

and as a consequence, if $F_0 \neq 0$ the equation $\mu_1' = \mu_2' + zJ\kappa\mu_2'$ has the unique solution

$$\mu_2' = \mu_1' - zJ\frac{F_1}{F_0}\mu_1'.$$

Since for every x of the class $\mathfrak{X}$ the function η belongs to the class $\mathfrak{K}$, we can set up these same determinants for η and obtain

$$F_{0\eta}(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} J_{p_1'' p_1'} \cdots J_{p_n'' p_n'} \eta \begin{pmatrix} p_1' & \cdots & p_n' \\ p_1'' & \cdots & p_1'' \end{pmatrix}; x \Big)$$

* Cf. E. H. Moore, *Integral equations*, loc. cit., pp. 353, 354.

and

$$F_{1\eta}(x) = \eta + \sum_{n=1}^{\infty} \frac{1}{n!} J_{p_1'' p_1'} \cdots J_{p_n'' p_n'}\, \eta \begin{pmatrix} p' & p_1' & \cdots & p_n' \\ p'' & p_1'' & \cdots & p_1'' \end{pmatrix}; x \biggr).$$

It can be shown that the series for $F_{0\eta}$ converges uniformly on $\mathfrak{X}$ and hence represents a continuous function on $\mathfrak{X}$; further that the series for $F_{1\eta}$ converges on $\mathfrak{P}' \mathfrak{P}'' \mathfrak{X}$ relatively uniformly as to the class $\mathfrak{H}$ and hence is again a function of this class. In case η belongs to the class $\mathfrak{H}'$ we can go a step further and assert the

THEOREM. *If η is of class $\mathfrak{H}'$ then $F_0(x)$ is of class $\mathfrak{C}'$, and we have*

$$D_x F_0(x) = F_0 JD\eta - J_{41} J_{23}(D\eta) F_1.$$

By following the line of reasoning used by Fredholm* in the derivation of an analogous equation, we obtain the above expression for $D_x F_0(x)$, and conclude therefrom that $F_0(x)$ is of class $\mathfrak{C}'$.

A well-known consequence of this result is the

THEOREM. *If κ_1 and κ_2 belong to the class $\mathfrak{K}$ then the Fredholm determinant of $\kappa_1 + \kappa_2 + J\kappa_1 \kappa_2$ is equal to the product of the determinants for κ_1 and κ_2, i. e.,*

$$F_0(\kappa_1 + \kappa_2 + J\kappa_1 \kappa_2) = F_0(\kappa_1) F_0(\kappa_2).$$

Since η_1 and η_2 of class $\mathfrak{H}$ for fixed x belong to $\mathfrak{K}$ it follows that

$$F_0(\eta_1 + \eta_2 + J\eta_1 \eta_2) = F_0(\eta_1) F_0(\eta_2).$$

9. **The Fredholm determinant of η_0.** Since the function η_0, the solution of equation (A) for which $\eta(x_0) = 0$, whose existence was shown in § 7, is of class $\mathfrak{H}'$, we can construct the Fredholm determinant of this function and apply the results of the preceding paragraph. We obtain

$$DF_{0\eta_0} = F_{0\eta_0} JD\eta_0 - J_{41} J_{23}(D\eta_0) F_{1\eta_0}.$$

But

$$D\eta_0 = \alpha + J\alpha\eta_0.$$

Substituting and making use of the commutativity of successive J-operations, we get

$$DF_{0\eta_0} = F_{0\eta_0} J\alpha - JJ\alpha(\eta_0 F_{0\eta_0} - F_{1\eta_0} - J\eta_0 F_{1\eta_0}).$$

But η_0, $F_{0\eta_0}$, and $F_{1\eta_0}$ are connected by the relation:

$$\eta_0 F_{0\eta_0} - F_{1\eta_0} - J\eta_0 F_{1\eta_0} = 0.$$

Hence

$$DF_{0\eta_0} = F_{0\eta_0} J\alpha.$$

Now $F_{0\eta_0}$ is not identically zero in x, since, for $x = x_0$ it has the value unity.

* *Sur une classe d'équations fonctionelles.* Acta Mathematica, vol. 27 (1903), pp. 379, 380.

Hence for the values of x for which $F_{0\eta_0}$ is not zero, we have

$$\frac{DF_{0\eta_0}}{F_{0\eta_0}} = J\alpha, \qquad \text{i. e.,} \qquad F_{0\eta_0} = e^{IJ\alpha}.$$

We have thus proved

THEOREM II. *The Fredholm determinant of the solution η_0 of equation (A) is not zero. It has the value $e^{IJ\alpha}$.*

10. **The General solution of equations (A) and (B).** By following the line of reasoning indicated by Schlesinger,* we show that any solution η of class $\mathfrak{H}'$ of equation (A) can be expressed in the form

$$\eta = \eta_1 + \eta_0 + J\eta_0\,\eta_1,$$

and thence that $D\eta_1 = 0$; that is, η_1 is independent of x and hence a function of the class $\mathfrak{K}$. We therefore have:

THEOREM III*a*. *The general solution of equation (A) of the class $\mathfrak{H}'$ can be written in the form*

$$\eta = \kappa + \eta_0 + J\eta_0\,\kappa,$$

where κ is a function of the class $\mathfrak{K}$, and $\eta_0(x_0) = 0$.

From the second theorem of § 8 it follows that the Fredholm determinant of η is the product of the Fredholm determinants of κ and η_0 and hence is zero only when the determinant of κ is zero. Moreover the form of the general solution together with the fact that the Fredholm determinant of η_0 is not zero yields the more general

EXISTENCE THEOREM. *There exists a unique solution of equation (A) of class $\mathfrak{H}'$ which satisfies the initial condition $\eta(x_0) = \kappa_0$, where x_0 is any element of $\mathfrak{X}$ and κ_0 any function of $\mathfrak{K}$.*

We can proceed in a somewhat similar way for the homogeneous equation (B). By observing that the difference of two solutions of the equation (A) is a solution of equation (B) we easily obtain†

THEOREM III*b*. *The general solution of $D\eta = J\alpha\eta$ can be written in the form*

$$\eta = \kappa + J\eta_0\,\kappa,$$

where κ is a function of the class $\mathfrak{K}$.

Observe that this theorem does not give the general solution of equation (B) in terms of a particular solution of the same equation, but in terms of a particular solution of the non-homogeneous equation (A). As in the previous case we can state the

EXISTENCE THEOREM. *There exists a unique solution of equation (B) of class $\mathfrak{H}'$ which satisfies the initial condition $\eta(x_0) = \kappa_0$, where x_0 is any element of $\mathfrak{X}$ and κ_0 is any function of $\mathfrak{K}$.*

* Loc. cit., pp. 99, 100.

† Cf. Schlesinger, loc. cit., p. 97.

11. **The adjoint equations.** In the theory of systems of n linear differential equations, there is associated with the system

$$Dy_i(x) = \sum_{j=1}^{n} \alpha_{ij}(x)\, y_j(x)$$

the system

$$Dz_i(x) = -\sum_{j=1}^{n} z_j(x)\, \alpha_{ji}(x)$$

obtained by changing the sign of the functions $\alpha_{ij}(x)$ and summing with respect to the first of the two subscripts instead of the second. Similarly, in the case where i and j are replaced by continuous variables, Schlesinger* points out that it is desirable to call

$$D\eta(p, q; x) = -\alpha(p, q; x) - \int_a^b \eta(p, r; x)\, \alpha(r, q; x)\, dr$$

the adjoint of the equation

$$D\eta(p, q; x) = \alpha(p, q; x) + \int_a^b a(p, r; x)\, \eta(r, q; x)\, dr.$$

On the basis of these observations we define the equation

$$(A') \qquad D\eta = -\alpha - J\eta\alpha$$

to be the adjoint of equation (A), and

$$(B') \qquad D\eta = -J\eta\alpha$$

to be the adjoint of equation (B).

If η and $\hat{\eta}$ are any functions of $\mathfrak{H}'$, we denote by $M_1(\eta)$ the function of $\mathfrak{H}$ represented by the expression $D\eta - \alpha - J\alpha\eta$, and by $N_1(\hat{\eta})$ the function represented by $D\hat{\eta} + \alpha + J\hat{\eta}\alpha$. We then have the following formula which is analogous to the *Green's Formula*,

$$(G_1) \quad D(\eta + \hat{\eta} + J\hat{\eta}\eta) = M_1(\eta) + N_1(\hat{\eta}) + J\hat{\eta}M_1(\eta) + JN_1(\hat{\eta})\eta$$

or

$$(\eta + \hat{\eta} + J\hat{\eta}\eta)_{x_0} = I\Big(M_1(\eta) + N_1(\hat{\eta}) + J\hat{\eta}M_1(\eta) + JN_1(\hat{\eta})\eta\Big).$$

For by definition we have

$$\begin{aligned} J\Big(\hat{\eta}M_1(\eta) + N_1(\hat{\eta})\eta\Big) &= J\Big(\hat{\eta}D\eta + (D\hat{\eta})\eta + \alpha\eta - \hat{\eta}\alpha - \hat{\eta}J\alpha\eta + (J\hat{\eta}\alpha)\eta\Big) \\ &= J(D\hat{\eta}\eta) + J\alpha\eta - J\hat{\eta}\alpha \\ &= J(D\hat{\eta}\eta) - M_1(\eta) - \alpha + D\eta - N_1(\hat{\eta}) + \alpha + D\hat{\eta} \end{aligned}$$

or

$$D\eta + D\hat{\eta} + JD\hat{\eta}\eta = M_1(\eta) + N_1(\hat{\eta}) + J\hat{\eta}M_1(\eta) + JN_1(\hat{\eta})\eta.$$

* Loc. cit., p. 117.

By using the commutativity of the operators D and J when operating on functions of the class $\mathfrak{H}'$ we get the first form given above. The second form is the result of integration between x_0 and x.

Similarly if we denote by $M_2(\eta)$ the expression $D\eta - J\alpha\eta$, and by $N_2(\hat{\eta})$ the function $D\hat{\eta} + J\hat{\eta}\alpha$, we have

$$(G_2) \qquad DJ\hat{\eta}\eta = J\left(\hat{\eta}M_2(\eta) + N_2(\hat{\eta})\eta\right)$$

or

$$(J\hat{\eta}\eta)_{x_0}^{x} = IJ\left(\hat{\eta}M_2(\eta) + N_2(\hat{\eta})\eta\right).$$

Formula G_1 is immediately applicable to the question of the solution of the equation (A'). As a matter of fact, suppose $\hat{\eta}_0$ is a solution of (A') for which $\hat{\eta}_0(x_0) = 0$. If we substitute for η the solution η_0 of (A) and for $\hat{\eta}$ this solution $\hat{\eta}_0$ of (A') we have

$$\eta_0 + \hat{\eta}_0 + J\hat{\eta}_0\,\eta_0 = 0.$$

On the other hand if $\hat{\eta}_0$ is related to η_0 by means of this last equation we have from formula (G_1)

$$M_1(\eta_0) + N_1(\hat{\eta}_0) + J\hat{\eta}_0\,M_1(\eta_0) + JN_1(\hat{\eta}_0)\eta_0$$
$$= N_1(\hat{\eta}_0) + JN_1(\hat{\eta}_0)\eta_0 = 0.$$

But since the Fredholm determinant of η_0 is not zero, this has but one solution

$$N_1(\hat{\eta}_0) = 0,$$

that is, we have

THEOREM IV. *The conditions* $N_1(\hat{\eta}_0) = 0$ *with* $\hat{\eta}_0(x_0) = 0$ *are equivalent to*

$$\eta_0 + \hat{\eta}_0 + J\hat{\eta}_0\,\eta_0 = 0.$$

However, this is in the form of a reciprocal relation in the theory of linear integral equations and therefore at once follows

COROLLARY I. *The solution of equation* (A') *which satisfies the initial condition* $\hat{\eta}_0(x_0) = 0$ *is*

$$\hat{\eta}_0 = -\frac{F_{1_{\eta_0}}}{F_{0_{\eta_0}}},$$

where $F_{0_{\eta_0}}$ *and* $F_{1_{\eta_0}}$ *are respectively the Fredholm determinant and first minor of* η_0.

Further since the Fredholm determinant of $\eta_1 + \eta_2 + J\eta_1\,\eta_2$ is the product of the Fredholm determinants of η_1 and η_2, we have

COROLLARY II. *The Fredholm determinant of the solution* $\hat{\eta}_0$, *for which* $\hat{\eta}_0(x_0) = 0$, *is not zero. It has the value* $e^{-IJ\alpha}$.

In a manner similar to that used for equations (A) and (B), we prove

THEOREM III′. *The general solution of equation* (A') *is expressible in the form*

$$\hat{\eta} = \hat{\eta}_0 + \kappa + J\kappa\hat{\eta}_0$$

and of equation (B'), *in the form*

$$\hat{\eta} = \kappa + J\kappa\hat{\eta}_0 .$$

For any solution of equation (A) and any solution of (A') we have

THEOREM IV′. *If* η *and* $\hat{\eta}$ *are any solutions of class* $\mathfrak{H}'$ *of equations* (A) *and* (A') *respectively, then*

$$\eta + \hat{\eta} + J\hat{\eta}\eta = \kappa ,$$

where κ *is some function of the class* $\mathfrak{K}$, *i. e., independent of* x.

In terms of the solution $\hat{\eta}_0$ of the equation (A') it is an easy matter* to obtain

$$\eta = \kappa + J\kappa\eta_0 + I\alpha_0 + J\eta_0\, I\alpha_0 + IJ\hat{\eta}_0\, \alpha_0 + J\eta_0\, IJ\hat{\eta}_0\, \alpha_0 ,$$

as the general solution for the non-homogeneous equation

$$(C) \qquad D\eta = \alpha_0 + J\alpha\eta .$$

In a similar way, the general solution of the equation

$$(C') \qquad D\eta = \alpha_0 - J\eta\alpha$$

is expressible in terms of the solution η_0 of equation (A).

12. **Some examples.** We digress in this paragraph in order to apply the results of the preceding paragraphs to some special cases.

We consider first of all the case in which the function α is independent of x, i. e., α is the same function of the class $\mathfrak{K}$ for every x of $\mathfrak{X}$. In the solution η_0 of the equation (A) for which $\eta_0(x_0) = 0$,

$$\eta_0 = \sum_{n=0}^{\infty} (IJ\alpha)^n I\alpha ,$$

it will then be possible to carry out the integrations and we obtain

$$\eta_0 = \sum_{n=1}^{\infty} \frac{(x - x_0)^n}{n!} \alpha^{(n)} ,$$

where $\alpha^{(n)}$ is defined as follows:

$$\alpha^{(n)} = J\alpha^{(n-1)}\,\alpha \qquad \text{and} \qquad \alpha^{(1)} = \alpha ,$$

i. e., it is the nth iterated kernel of α in the sense of linear integral equations. On account of the properties of the class $\mathfrak{K}$ and the operator J, for every function α there will exist nowhere negative functions μ_0' and μ_0'' and a constant b such that $\alpha^{(n)} \leqq \mu_0' \mu_0'' b^{n-1}$ as in § 7 above, or in the theory of integral equations. It follows then that the series for η_0 is convergent for all values of x, relatively uniformly as to the class $\mathfrak{K}$ on $\mathfrak{P}'\mathfrak{P}''$, so that η_0 is of class $\mathfrak{H}$ on $\mathfrak{P}'\mathfrak{P}''\mathfrak{X}$, where $\mathfrak{X}$ may be the class $-\infty < x < \infty$. The same will be

* Cf. Schlesinger, loc. cit., pp. 119, 120.

true of the function obtained by putting $x_0 = 0$. This function we denote by $\eta(\alpha x)$, i. e.,

$$\eta(\alpha x) = \sum_{n=1}^{\infty} \frac{x^n \alpha^{(n)}}{n!} = \sum_{n=1}^{\infty} \frac{(x\alpha)^{(n)}}{n!}.$$

In the case in which

$$\mathfrak{P}' = \mathfrak{P}'' = \mathfrak{P}^{\mathrm{IV}} = [0 \leqq p \leqq 1]$$

and $\mathfrak{M}' = \mathfrak{M}'' = \mathfrak{M}^{\mathrm{IV}}$ = the class of all continuous functions on $\mathfrak{P}^{\mathrm{IV}}$, Schlesinger has called the function $\eta(\alpha x)$ the *Volterra Transcendental.**

An addition theorem for this function η is an immediate consequence of the form of the general solution of the equation (A). For evidently

$$\eta_0(x) = \eta[\alpha(x - x_0)],$$

and hence

$$\eta(\alpha x) = \kappa + \eta[\alpha(x - x_0)] + J\eta[\alpha(x - x_0)]\kappa.$$

The function κ is determined by letting $x = x_0$. This gives $\kappa = \eta(\alpha x_0)$, i. e.,

$$\eta(\alpha x) = \eta(\alpha x_0) + \eta[\alpha(x - x_0)] + J\eta[\alpha(x - x_0)]\eta(\alpha x_0).$$

If we let $x = y + z$ and $x_0 = y$, then

$$\eta[\alpha(y + z)] = \eta(\alpha y) + \eta(\alpha z) + J\eta(\alpha z)\eta(\alpha y).$$

Since $\eta(\alpha 0) = 0$, it follows from this that

$$\eta(\alpha y) + \eta(\alpha(-y)) + J\eta[\alpha(-y)]\eta(\alpha y) = 0,$$

i. e., $\eta(\alpha(-y))$ is the reciprocal of $\eta(\alpha y)$, a result which we might expect if we note that the solution of the adjoint equation $D\eta = -\alpha - J\eta\alpha$ for which $\eta(0) = 0$ is

$$\hat{\eta}(x) = \sum_{n=1}^{\infty} \frac{(-\alpha)^{(n)} x^n}{n!} = \sum_{n=1}^{\infty} \frac{\alpha^{(n)}(-x)^n}{n!} = \eta[\alpha(-x)].$$

The function $\eta(\alpha x)$ thus has properties somewhat similar to those of e^x. As a matter of fact, we can represent $\eta(\alpha x)$ symbolically in the form

$$(e^{\alpha x} - 1),$$

where α^n is to be replaced by $\alpha^{(n)}$. If we note that

$$(e^{\alpha_1 x} - 1)(e^{\alpha_2 y} - 1) = J\eta(\alpha_1 x)\eta(\alpha_2 y),$$

we obtain from

$$\begin{aligned}(e^{\alpha_1 x} - 1)(e^{\alpha_2 y} - 1) &= e^{\alpha_1 x + \alpha_2 y} - e^{\alpha_1 x} - e^{\alpha_2 y} + 1 \\ &= (e^{\alpha_1 x + \alpha_2 y} - 1) - (e^{\alpha_1 x} - 1) - (e^{\alpha_2 y} - 1)\end{aligned}$$

* Cf. Schlesinger, loc. cit., p. 113; Volterra, *Leçons sur les fonctions de lignes*, pp. 127, 158, and 159.

the more general addition theorem

$$J\eta(\alpha_1 x)\eta(\alpha_2 y) = \eta(\alpha_1 x + \alpha_2 y) - \eta(\alpha_1 x) - \eta(\alpha_2 y),$$

provided α_1 and α_2 are such that

$$J\alpha_1 \alpha_2 = J\alpha_2 \alpha_1,$$

i. e., α_1 and α_2 are *permutable.**

Another type of interesting results may be obtained by specializing the ranges $\mathfrak{P}'$ and $\mathfrak{P}''$, the classes of functions $\mathfrak{M}'$ and $\mathfrak{M}''$, and the operator J. We note the following intances:

(*a*) *Linear integro-differential equations.*† We take $\mathfrak{P}' = \mathfrak{P}'' = \mathfrak{P}^{\mathrm{IV}} =$ the class of all points on the linear interval $0 \leqq p \leqq 1$. Further let $\mathfrak{M}' = \mathfrak{M}'' = \mathfrak{M}^{\mathrm{IV}} =$ the class of all continuous function on $\mathfrak{P}^{\mathrm{IV}}$. Then $\mathfrak{K}$ is the class of all continuous functions on $\mathfrak{P}\mathfrak{P}$ and $\mathfrak{H}$ is the class of all continuous functions on $\mathfrak{P}\mathfrak{P}\mathfrak{X}$.‡ The operator J we assume to be the definite integral $\int_0^1 dp$. Then our differential equation (A) takes the form

$$D_x \eta(pqx) = \alpha(pqx) + \int_0^1 \alpha(prx)\eta(rqx)\,dr.$$

We conclude at once, then, that there exists a unique solution of this equation, $\eta_0(pqx)$, for which $\eta(x_0) = 0$. Moreover if we build the ordinary Fredholm determinant for this solution it will not be zero on $\mathfrak{X}$ but will have as its value

$$e^{I\int_0^1 \alpha(ppx)dp}.$$

(*b*) *Systems of linear integro-differential equations.*§ Let $\mathfrak{P}' = \mathfrak{P}'' =$ the composite class $\mathfrak{P}^{\mathrm{II}_n}\mathfrak{P}^{\mathrm{IV}} =$ class of all pairs (ip), where $i = 1, \cdots, n$, and $0 \leqq p \leqq 1$. We assume that $\mathfrak{M}' = \mathfrak{M}'' = (\mathfrak{M}^{\mathrm{II}_n}\mathfrak{M}^{\mathrm{IV}})_* =$ the class of all sets of n continuous functions on $\mathfrak{P}^{\mathrm{IV}}$. Then $\mathfrak{K}$ will be the class of all sets of n^2 continuous functions of two variables on $\mathfrak{P}^{\mathrm{IV}}\mathfrak{P}^{\mathrm{IV}}$ and $\mathfrak{H}$ will be the class of all sets of n^2 continuous functions of three variables on $\mathfrak{P}^{\mathrm{IV}}\mathfrak{P}^{\mathrm{IV}}\mathfrak{X}$. We take for the operator J the bipartite operator $\sum_{i=1}^{i=n}\int_0^1$. Then we have for consideration the following system of linear integro-differential equations

$$D_x \eta_{ij}(pqx) = \alpha_{ij}(pqx) + \sum_{k=1}^{n}\int_0^1 \alpha_{ik}(prx)\eta_{kj}(rqx)\,dr.$$

As a consequence of our general existence theorem, this equation has a

* Cf. Volterra, loc. cit., pp. 124, 158 and 159.

† Cf. Schlesinger, loc. cit., pp. 84 ff. He considers the case in which the variable x is replaced by the complex variable z. The functions η are analytic in z.

‡ Cf. for instance O. Bolza, loc. cit., p. 291.

§ See also § 12 below.

unique system of solutions for which $\eta_{ij}(pqx_0) = 0$. The Fredholm determinant of this system of solutions which can be written in the form

$$1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} \int_0^1 \cdot \int_0^1 \begin{vmatrix} \eta_{i_1 i_1}(x_1 x_1) & \cdots & \eta_{i_1 i_m}(x_1 x_m) \\ \cdot & \cdot & \cdot \\ \eta_{i_m i_1}(x_m x_1) & \cdots & \eta_{i_m i_m}(x_m x_m) \end{vmatrix} dx_1 \cdots dx_m$$

has the value

$$e^{\sum_{i=1}^{n} \int_0^1 I\alpha_{ii}(ppx)dp}.$$

The general solution of the system will be:

$$\eta_{ij}(pqx) = \kappa_{ij}(pq) + \eta_{0ij}(pqx) + \sum_{k=1}^{n} \int_0^1 \eta_{0ik}(prx)\kappa_{kj}(rq)\,dr,$$

where $\kappa_{ij}(pq)$ are a system of continuous functions on $\mathfrak{P}\mathfrak{P}$.

(c) *Infinite systems of differential equations.* Let $\mathfrak{P}' = \mathfrak{P}'' = \mathfrak{P}^{\mathrm{III}}$, the class of all positive integers, i. e., $p = 1, \cdots n \cdots$. We assume that $\mathfrak{M}' = \mathfrak{M}^{\mathrm{III}_f}$ = the class of all functions μ' for which $\sum_p |\mu_p|^f$ is convergent, f being greater than unity, and $\mathfrak{M}'' = \mathfrak{M}^{\mathrm{III}_{f/(f-1)}}$ = the class of all functions μ'' for which

$$\sum_p |\mu_p''|^{f/(f-1)}$$

is convergent. Then $\mathfrak{K}$ is the class of all functions $\kappa(pq)$ for which there exist μ_p' and μ_q'' so that

$$|\kappa(pq)| \leqq \mu'(p)\mu''(q).*$$

Further $\mathfrak{H}$ will be the class of all functions which are continuous in x, for p and q fixed, and are dominated by a function of the class $\mathfrak{K}$, i. e., if $\eta(pqx)$ belongs to $\mathfrak{H}$, it will be continuous in x, and there will exist a $\kappa(pq)$ such that $|\eta(pqx)| \leqq |\kappa(pq)|$, and conversely.† We assume that the operator J on $\mathfrak{K}$ is the $\sum_{p=1}^{p=\infty} \kappa(pp)$. That this series actually converges is a result of the inequality:‡

$$[|\sum_{p=1}^{\infty} \mu'(p)\mu''(p)|]^f \leqq [\sum_{p=1}^{\infty} |\mu'(p)|^f][\sum_{p=1}^{\infty} |\mu''(p)|^{f/(f-1)}]^{f-1}.$$

Since $\Sigma|\mu'(p)|^f$ and $\Sigma|\mu''(p)|^{f/(f-1)}$ converge and there exist $\mu'(p)$ and $\mu''(p)$ such that $|\kappa(pp)| \leqq \mu'(p)\mu''(p)$, it follows that $\Sigma\kappa(pp)$ will converge also. Evidently $\sum_{p=1}^{p=\infty}$ has the properties L and M. We can then

* Cf. for instance Bolza, loc. cit., p. 292.

† Cf. Moore, *General Analysis*, pp. 110–114, 146, 147. This class may also be defined to be the class of all functions which belong to $\mathfrak{K}$ for every x and are continuous in x relatively uniformly as to the class $\mathfrak{K}$. See p. 101.

‡ Cf. Riesz, *Theorie d'équations linéaires à une infinité d'inconnus*, p. 45.

write our differential equation

$$D\eta(pqx) = \alpha(pqx) + \sum_{r=1}^{\infty} \alpha(prx)\eta(rqx), \qquad p, q = 1, 2, \cdots, n, \cdots,$$

and conclude that this infinite system has a unique solution of class $\mathfrak{H}'$, for which $\eta_0(pqx_0) = 0$. The Fredholm determinant of this solution reduces to the convergent infinite determinant of

$$\delta(pq) + \eta_0(pqx),$$

where $\delta(pq)$ is the Kronecker δ, and this determinant has the value

$$e^{I\,\Sigma_{p=1}^{p=\infty}\alpha(ppx)}.$$

The general solution of the equation of class $\mathfrak{H}'$ is

$$\eta(pqx) = \kappa(pq) + \eta_0(pqx) + \sum_r \eta_0(prx)\kappa(rq),$$

while the general solution of the associated homogeneous equation is

$$\eta(pqx) = \kappa(pq) + \sum_r \eta_0(prx)\kappa(rq),$$

where κ is any function of the class $\mathfrak{K}$. If in this last case we consider q fixed, we get a function $\phi(px)$ which belongs to $\mathfrak{M}'$ for every x. We can consider it a solution of the equation

$$D\phi(px) = \sum_r \alpha(prx)\phi(rx).$$

If on the other hand we construct the adjoint equation

$$D\eta(pqx) = -\sum_r \eta(prx)\alpha(rqx)$$

and fix p in the general solution, we get a function $\psi(qx)$ which belongs to $\mathfrak{M}''$ for every x, and can be considered a solution of the equation

$$D\psi(px) = -\sum_r \psi(rx)\alpha(rpx).$$

We have here the same phenomenon which we find in the theory of equations of infinitely many variables, where solutions of adjoint equations belong to $\mathfrak{M}^{\mathrm{III}_f}$ and $\mathfrak{M}^{\mathrm{III}_{f/(f-1)}}$ respectively.

(*d*) Finally we consider an instance in which the ranges $\mathfrak{P}'$ and $\mathfrak{P}''$ are different. Let $\mathfrak{P}' = \mathfrak{P}^{\mathrm{II}_n}$ = class $i = 1, 2, \cdots, n$, and $\mathfrak{P}'' = \mathfrak{P}^{\mathrm{IV}}$ = interval $0 \leqq p \leqq 1$. Let $\mathfrak{M}' = \mathfrak{M}^{\mathrm{II}_n}$ = class of all n-partite numbers, and $\mathfrak{M}'' = \mathfrak{M}^{\mathrm{IV}}$ = class of all continuous functions on $\mathfrak{P}^{\mathrm{IV}}$. Then $\mathfrak{K}$ will be the class of all sets of n continuous functions on $\mathfrak{P}$, while $\mathfrak{H}$ will be the class of all sets of n continuous functions on $\mathfrak{P}\mathfrak{X}$. Finally let the operator J on a

function $\kappa(i, p)$, or $\kappa_i(p)$ be

$$\sum_{i=1}^{n} \int_0^1 \omega_i(p)\kappa_i(p)\,dp,$$

where $\omega_i(p)$ belongs to $\mathfrak{K}$.

Then our differential equation becomes

$$D\eta_i(px) = \alpha_i(px) + \sum_{j=1}^{n} \int_0^1 \alpha_i(qx)\omega_j(q)\eta_j(px)\,dq.$$

We observe that every solution of this equation will also be a solution of the linear integro-differential equation

$$D\eta(pqx) = \alpha(pqx) + \int_0^1 \alpha(prx)\eta(rqx)\,dr,$$

where

$$\eta(pqx) = \sum_{i=1}^{n} \omega_i(p)\eta_i(qx)$$

and

$$\alpha(pqx) = \sum_{i=1}^{n} \omega_i(p)\alpha_i(qx),$$

and also of the linear system of equations

$$D\eta(ijx) = \alpha(ijx) + \sum_{k=1}^{n} \alpha(ikx)\eta(kjx),$$

where

$$\eta(ijx) = \int_0^1 \eta_i(px)\omega_j(p)\,dp$$

and

$$\alpha(ijx) = \int_0^1 \alpha_i(px)\omega_j(p)\,dp.$$

The converse is not true, excepting when ω satisfies certain conditions.

Applying our general existence theorem, we conclude that there will exist for our equation a unique solution $\eta_i(px)$ for which $\eta_i(px_0) = 0$. When we build the expression corresponding to the Fredholm determinant of this solution with the operator defined above, we find that it may be regarded as the Fredholm determinant of $\eta(pqx) = \sum_{i=1}^{i=n} \omega_i(p)\eta_i(qx)$, or as the ordinary determinant of $\delta(ij) + \eta(ij)$ where

$$\eta(ij) = \int_0^1 \eta_i(px)\omega_j(p)\,dp.$$

Its value will be

$$e^{I\int_0^1 \eta(ppx)dp} = e^{I\sum_{i=1}^{n}\eta(iix)} = e^{I\int_0^1 \sum_{i=1}^{n}\alpha_i(px)\omega_i(p)dp}.$$

13. **On systems of linear differential equations.** The results of §§ 6–10

can be easily extended to the case of a system of equations. As a matter of fact a simple transformation reduces the consideration of a system to that of a single equation.

Suppose then that in place of the class $\mathfrak{P}'$ we have n classes $\mathfrak{P}'_i$ $(i = 1, \cdots, n)$, and instead of the class $\mathfrak{P}''$ we have n classes $\mathfrak{P}''_i$ $(i = 1, \cdots, n)$ which are not necessarily all distinct. Suppose further that in place of $\mathfrak{M}'$ on $\mathfrak{P}'$ to $\mathfrak{A}$ we have n classes $\mathfrak{M}'_i$ on $\mathfrak{P}'_i$ to $\mathfrak{A}$, and in place of $\mathfrak{M}''$ on $\mathfrak{P}''$ to $\mathfrak{A}$ we have n classes $\mathfrak{M}''_i$ on $\mathfrak{P}''_i$ to $\mathfrak{A}$, concerning each of which it will be assumed that they have the properties $LCDD_0$. From these classes we construct the n^2 classes $\mathfrak{K}_{ij} = (\mathfrak{M}'_i \mathfrak{M}''_j)_*$ on $\mathfrak{P}'_i \mathfrak{P}''_j$ to $\mathfrak{A}$, and the n^2 classes $\mathfrak{H}_{ij} = (\mathfrak{M}'_i \mathfrak{M}''_j \mathfrak{C})_*$ on $\mathfrak{P}'_i \mathfrak{P}''_j \mathfrak{X}$ to $\mathfrak{A}$. We replace finally the operator J on $\breve{\mathfrak{K}}$ to $\mathfrak{A}$ by the n operators J_i on $\breve{\mathfrak{K}}_{ii}$ to $\mathfrak{A}$, each of which will be supposed to have the properties L and M. We consider then in place of the equation (A) the system of equations

$$(A_n) \qquad D\eta_{ij} = \alpha_{ij} + \sum_{k=1}^{n} J_k\, \alpha_{ik}\, \eta_{kj}$$

and the homogeneous system associated with it

$$(B_n) \qquad D\eta_{ij} = \sum_{k=1}^{n} J_k\, \alpha_{ik}\, \eta_{kj}.$$

In treating the corresponding extension in the general theory of linear integral equations, Moore* has suggested a process, viz., that of *adjunctional composition* which reduces the system of equations to a single equation. This same process is applicable here. We assume that the classes $\mathfrak{P}'_i$ have no elements in common, a situation which can always be attained by a suitable transformation. We make the same assumption in the case of the classes $\mathfrak{P}''_j$. Then we define the class $\mathfrak{P}'$ to be the class of all elements belonging to any of the $\mathfrak{P}'_i$, i. e., $\mathfrak{P}' = \sum_{i=1}^{i=n} \mathfrak{P}'_i$, and $\mathfrak{P}''$ to be the class of all elements belonging to any of the $\mathfrak{P}''_j$, i. e. $\mathfrak{P}'' = \sum_{j=1}^{j=n} \mathfrak{P}''_j$. The class $\mathfrak{P}'\mathfrak{P}''$ will be the totality of elements in $\mathfrak{P}'_i \mathfrak{P}''_j$. The class $\mathfrak{K}$ will consist of all functions κ for which we have

$$\kappa(p'_i\, p''_j) = \kappa_{ij}(p'_i\, p''_j)$$

and the class $\mathfrak{H}$ will consist of all functions for which we have

$$\eta(p'_i\, p''_j\, x) = \eta_{ij}(p'_i\, p''_j\, x).$$

Further we define J operating on a function $\breve{\kappa}$ to be

$$J\breve{\kappa} = \sum_{i=1}^{n} J_i\, \breve{\kappa}_{ii}.$$

Then the classes $\mathfrak{K}$ and $\mathfrak{H}$ and the operator J will have the properties enum-

* Cf. E. H. Moore, *Integral Equations*, loc. cit., pp. 355–357.

erated in §§ 3–5, and our system of differential equations reduces to the single equation

$$D\eta = \alpha + J\alpha\eta,$$

while the system (B_n) becomes

$$D\eta = J\alpha\eta.$$

By applying the theorems of §§ 6–10, and replacing α, η, and J by the expressions from which they were derived, we get results of the following type:

THEOREM I. *The system of equations (A_n) has a unique set of solutions η_{ij} which satisfy the initial conditions $\eta_{ij}(x_0) = 0$.*

THEOREM II. *The Fredholm determinant of this set of solutions is not zero on $\mathfrak{X}$. It has the value*

$$e^{I \sum_{i=1}^{i=n} J_i \alpha_{ii}}.$$

The Fredholm determinant of a set of functions η_{ij} would take the form

$$1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} J_{i_1} \cdots J_{i_m} \begin{vmatrix} \eta_{i_1 i_1}(p'^{(1)}_{i_1}\, p''^{(1)}_{i_1}) \cdots \eta_{i_1 i_m}(p'^{(1)}_{i_m}\, p''^{(m)}_{i_m}) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \eta_{i_m i_1}(p'^{(m)}_{i_m}\, p''^{(1)}_{i_1}) \cdots \eta_{i_m i_m}(p'^{(m)}_{i_m}\, p''^{(m)}_{i_1}) \end{vmatrix}.$$

THEOREM III. *The general solution of the system (A_n) is*

$$\eta_{ij} = \kappa_{ij} + \eta_{0ij} + \sum_{k=1}^{n} J_k\, \eta_{0ik}\, \kappa_{kj},$$

where κ_{ij} is any function of $\mathfrak{K}_{ij}$ and η_{0ij} is the solution for which

$$\eta_{0ij}(x_0) = 0.$$

The general solution of the associated system (B_n) is

$$\eta_{ij} = \kappa_{ij} + \sum_{k=1}^{n} J_k\, \eta_{0ik}\, \kappa_{kj}.$$

The adjoint equations of (A_n) and (B_n) are

$$(A'_n) \qquad D\eta_{ij} = -\alpha_{ij} - \sum_{k=1}^{n} J_k\, \eta_{ik}\, \alpha_{kj}$$

and

$$(B'_n) \qquad D\eta_{ij} = -\sum_{k=1}^{n} J_k\, \eta_{ik}\, \alpha_{kj}.$$

If we denote by $\bar{\eta}_{0ij}$ the set of solutions of equations (A'_n) for which

$$\bar{\eta}_{0ij}(x_0) = 0,$$

then follows

THEOREM IV. *The functions η_{0ij} and $\bar{\eta}_{0ij}$ satisfy the reciprocal relations*

$$\eta_{0ij} + \bar{\eta}_{0ij} + \sum_{k=1}^{n} J_k\, \bar{\eta}_{0ik}\, \eta_{0kj} = 0.$$

If we denote by $M_{2ij}(\eta)$ the functions of $\mathfrak{H}_{ij}$ obtained by substituting a set of functions η_{ij} in the expression

$$D\eta_{ij} - \sum_{k=1}^{n} J_k\, \alpha_{ik}\, \eta_{kj} = M_{2ij}(\eta),$$

and similarly let

$$D\hat{\eta}_{ij} + \sum_{k=1}^{n} J_k\, \hat{\eta}_{ik}\, \alpha_{kj} = N_{2ij}(\hat{\eta}),$$

then we get the *Green's Formula*:

$$I \sum_{k=1}^{n} J_k \left(\hat{\eta}_{ik}\, M_{2kj}(\eta) + N_{2ik}(\hat{\eta})\, \eta_{kj} \right) = \left(\sum_{k=1}^{n} J_k\, \hat{\eta}_{ik}\, \eta_{kj} \right)_{x=x_0}^{x=x}.$$

14. **Mixed integro-differential equations.** An interesting application of the results of the previous paragraph is to the case of mixed differential equations.* Suppose that instead of the single operator J on $\breve{\kappa}$ we have n operators J_i on $\breve{\kappa}$. Then we might consider the equation

$$(B_1) \qquad D\eta = \sum_{i=1}^{n} J_i\, \alpha_i\, \eta.$$

If we assume that $\eta_{ij} = \eta$ and $\alpha_{ij} = \alpha_j$ for every i, then this equation repeated n times could be considered as a system (B_n)

$$D\eta_{ij} = \sum_{k=1}^{n} J_k\, \alpha_{ik}\, \eta_{kj},$$

and as a consequence the associated system would take the form

$$(A_1) \qquad D\eta_{ij} = \alpha_{ij} + \sum_{k=1}^{n} J_k\, \alpha_{ik}\, \eta_{kj}.$$

The existence theorem applied to this equation reads:

THEOREM I. *There exists a unique set of solutions of equations (A_1) which are such that $\eta_{0ij}(x_0) = 0$. On account of the fact that the system (A_1) really contains only n distinct equations there will be at most n distinct solutions η_{0j}.*

THEOREM II. *The Fredholm determinant of the η_{0j} which has the form:*

$$1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{i_1=1}^{n} J_{i_1} \cdots \sum_{i_m=1}^{n} J_{i_m} \begin{vmatrix} \eta_{i_1}(p_1' p_1'') & \cdots & \eta_{i_1}(p_1' p_m'') \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \eta_{i_m}(p_m' p_1'') & \cdots & \eta_{i_m}(p_m' p_m'') \end{vmatrix}$$

is not zero. Its value is $e^{I \sum_{i=1}^{i=n} J_i \alpha_i}$.

THEOREM III. *The general solution of equation (B_1) is*

$$\eta = \kappa + \sum_{i=1}^{n} J_i\, \eta_{0i}\, \kappa.$$

* Cf. E. H. Moore, *Integral Equations*, loc. cit., p. 358.

The adjoint of the system (A) would be:

$$D\eta_{ij} = -\alpha_{ij} - \sum_{k=1}^{n} J_k \, \eta_{ik} \, \alpha_{kj} .$$

Since however $\alpha_{ij} = \alpha_j$ for every i, we could drop the first subscripts and get the system of n equations

$$(A_1') \qquad D\eta_j = -\alpha_j - \sum_{k=1}^{n} J_k \, \eta_k \, \alpha_j .$$

It follows that the adjoint of the single equation (B_1) is the system*

$$(B_1') \qquad D\eta_j = -\sum_{k=1}^{n} J_k \, \eta_k \, \alpha_j .$$

THEOREM IV. *Between the solutions η_{0i} of the equation (A_1) and $\bar{\eta}_{0j}$ of the equations (A_1') for which $\eta_{0i}(x_0) = 0$ and $\bar{\eta}_{0j}(x_0) = 0$, we have the n relations*

$$\eta_{0i} + \bar{\eta}_{0i} + \sum_{k=1}^{n} J_k \, \bar{\eta}_{0k} \, \eta_{0i} = 0 \qquad (i = 1, 2, \cdots, n).$$

If we let

$$M(\eta) = D\eta - \sum_{i=1}^{n} J_i \, \alpha_i \, \eta$$

and

$$N_i(\bar{\eta}) = D\bar{\eta}_i + \sum_{j=1}^{n} J_j \, \bar{\eta}_j \, \alpha_i$$

we have the *Green's Formula*:

$$I \sum_{i=1}^{n} J_i \left(\bar{\eta}_i \, M(\eta) + N_i(\bar{\eta}) \, \eta \right) = \left(\sum_{i=1}^{n} J_i \, \bar{\eta}_i \, \eta \right)_{x=x_0}^{x=x} .$$

We obtain further interesting results if we apply the theorems of this paragraph to the mixed linear integro-differential equation

$$D\eta(pqx) = \sum_{i=1}^{n} \alpha_i(px) \, \eta(p_i qx) + \int_0^1 \alpha(prx) \, \eta(rqx) \, dr ,$$

in which we suppose that the ranges $\mathfrak{P}' = \mathfrak{P}'' = \mathfrak{P}^{\mathrm{IV}} = [0 \leqq p \leqq 1]$, $\alpha_i(px)$ belong to the class of all continuous functions on $\mathfrak{P}\mathfrak{X}$, and $\alpha(pqx)$ to the class of continuous functions on $\mathfrak{P}\mathfrak{P}\mathfrak{X}$, while p_i are a set of special values of the range $\mathfrak{P}$.

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April, 1916

* Cf. W. A. Hurwitz, *Mixed linear integral equations of the first order*, these Transactions, vol. 16 (1915), p. 121, where a similar situation occurs. The above seems to show why the adjoint of a single mixed equation is a system.

TRANSFORMATIONS T OF CONJUGATE SYSTEMS OF CURVES ON A SURFACE*

BY

LUTHER PFAHLER EISENHART

When a surface S is referred to a conjugate system of lines, its point coördinates are solutions of a partial differential equation of the Laplace type, called *the point equation* for the given conjugate system. Throughout this paper we consider surfaces referred to conjugate systems, and hence we will use the symbol S to denote either the surface or the parametric system upon it, as the case may be. When the developables of a linear congruence G meet S in the parametric conjugate system, we say that G and S are conjugate to one another. A second surface S_1 conjugate to G is said to be in the relation of a transformation T to S, or to be a T transform of S. Darboux† has shown that each solution ϕ of the adjoint equation of the point equation of S determines a congruence G conjugate to S, and that each solution of this point equation determines for any of these congruences G a conjugate surface S_1. Hence each pair of function ϕ and θ determines a transformation T, and every such transformation is so determined. It is the purpose of this paper to develop the theory of these transformations.

It is shown that if S_1 and S_2 are two T transforms of S, there exist ∞^2 surfaces S_{12} each of which is in the relation of transformations T with S_1 and S_2, and the determination of these surfaces requires only quadratures. We have thus proved the existence of a theorem of permutability of transformations T, which includes a similar theorem for the transformations K of conjugate systems with equal invariants‡ (see § 9), just as the latter embraces as a particular case the theorem established by Bianchi§ for transformations D_m of isothermic surfaces. In § 12 we extend the theorem of permutability so as to be concerned with eight surfaces.

When the function θ determining a transformation is a constant and the point coördinates are in the cartesian form, the corresponding tangent planes to S and S_1 are parallel, in which case we say that we have a parallel trans-

* Presented to the Society, Dec. 27, 1916.

† *Leçons*, vol. 2, pp. 225, 227.

‡ Cf. Eisenhart, These Transactions, vol. 15 (1914), pp. 404–8. Hereafter this memoir will be referred to as M_1.

§ Annali di Matematica, ser. 3, vol. 11 (1905), pp. 93–158.

formation. This result and the consideration of the relation between two transformations T determined by the same ϕ but different functions θ lead to results formerly found by the author* for certain types of transformations T and later by Jonas,† and enable us to put the equations of a general transformation T in another convenient form.

In § 9 we consider in particular the case where the point equation of S has equal invariants and its transforms possess the same property. The resulting transformations are the transformations‡ K previously studied by us in their relation to the transformations of Moutard of differential equations. As there shown, these transformations K include the transformations D_m of isothermic surfaces discovered by Darboux.§

Transformations T can be treated analytically also in terms of the tangential coördinates of the surface. This is done in § 10, and the relations between the two sets of equations are determined. In particular, the case where the tangential equation has equal invariants is studied, with the result that we are led to the transformations Ω previously discovered by the author.||

If x, y, z are the cartesian coördinates of a surface S and ω is any solution of the point equation of S, the surface $\bar{S}$ whose cartesian coördinates are $x/\omega, y/\omega, z/\omega$ is referred to a conjugate system. We say that $\bar{S}$ is a radial transform of S. Combinations of radial and T transformations are studied in § 13 in relation to the theorem of permutability of transformations T. In particular, it is shown that this theorem can be applied when a radial transformation is treated as a special type of transformation T.

1. Transformations T in homogeneous point coördinates

The necessary and sufficient condition that four functions, x, y, z, w, be the homogeneous point coördinates of a surface S, referred to a conjugate system of lines of parameters u and v, is that these functions satisfy an equation of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} + a\frac{\partial \theta}{\partial u} + b\frac{\partial \theta}{\partial v} + c\theta = 0, \tag{1}$$

where a, b, c are functions of u and v. We refer to this equation as the point equation of the conjugate system.

If the developables of a rectilinear congruence meet S in the parametric

*Rendiconti di Palermo, vol. 39 (1915), pp. 153–176. Hereafter this memoir will be referred to as M_2.

†Sitzungsberichte der Berliner Mathematischen Gesellschaft, vol. 14 (1915), pp. 96–118.

‡ M_1, pp. 397–430.

§Annales de l'école normale superieure, ser. 3, vol. 16 (1899), pp. 491–508.

|| M_2.

curves, the congruence and the parametric system are said to be conjugate to one another. Darboux* has shown that when a solution ϕ of the adjoint equation of (1), namely

$$\frac{\partial^2 \phi}{\partial u \partial v} - a\frac{\partial \phi}{\partial u} - b\frac{\partial \phi}{\partial v} + \left(c - \frac{\partial a}{\partial u} - \frac{\partial b}{\partial v}\right)\phi = 0, \tag{2}$$

is known, a congruence G_1 conjugate to the parametric system is given by quadratures. In fact, the point coördinates, x_1', y_1', z_1', w_1'; x_1'', y_1'', z_1'', w_1'' of the focal points F_1' and F_1'' of the congruence are given by expressions of the form

$$\begin{aligned} x_1' &= \int \phi_1\left(\frac{\partial x}{\partial u} + bx\right)du + x\left(\frac{\partial \phi_1}{\partial v} - a\phi_1\right)dv, \\ x_1'' &= \int x\left(\frac{\partial \phi_1}{\partial u} - b\phi_1\right)du + \phi_1\left(\frac{\partial x}{\partial v} + ax\right)dv. \end{aligned} \tag{3}$$

Furthermore, each solution of (1) leads by quadratures to another conjugate system conjugate to the above congruence. For, if θ_1 is a solution of (1), the function σ_1 given by

$$\sigma_1 = \int \phi_1\left(\frac{\partial \theta_1}{\partial u} + b\theta_1\right)du + \theta_1\left(\frac{\partial \phi_1}{\partial v} - a\phi_1\right)dv, \tag{4}$$

is a solution of the point equation of the surface (F_1'), the locus of F_1'. Hence by the theorem of Levy† the surface S_1, whose coördinates are given by equations of the form

$$x_1 = x_1' - \frac{\sigma_1}{\frac{\partial \sigma_1}{\partial v}}\frac{\partial x_1'}{\partial v} = x_1 - \frac{\sigma_1}{\theta_1}x, \tag{5}$$

is conjugate to the congruence G_1 whose focal surfaces are (F_1') and (F_1''). We say that S_1 is obtained from S by a transformation T.

The equations of S_1 can be given another form, if we look upon the lines of the congruence G_1 as tangent to the focal surface (F_1'') also. Evidently the function τ_1, given by

$$\tau_1 = \int \theta_1\left(\frac{\partial \phi_1}{\partial u} - b\phi_1\right)du + \phi_1\left(\frac{\partial \theta_1}{\partial v} + a\theta_1\right)dv, \tag{6}$$

is a solution of the point equation of (F_1''). Hence the equations of S_1 can be given the form

$$x_1 = -x_1'' + \frac{\tau_1}{\theta_1}x. \tag{7}$$

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* *Leçons*, vol. 2, p. 225.

† Journal de l'école polytechnique, cahier 56 (1886), p. 63; also Darboux, *Lecons*, vol. 2, p. 222.

From (4) and (6) we find the relation

$$\phi_1 \theta_1 = \sigma_1 + \tau_1. \tag{8}$$

From (5) or (7) we get by differentiation

$$\frac{\partial x_1}{\partial u} = \tau_1 \frac{\partial}{\partial u}\left(\frac{x}{\theta_1}\right), \qquad \frac{\partial x_1}{\partial v} = -\sigma_1 \frac{\partial}{\partial v}\left(\frac{x}{\theta_1}\right). \tag{9}$$

2. The inverse of a transformation T

It is readily found from (9) that x_1, y_1, z_1, w_1 satisfy the equation

$$\frac{\partial^2 \theta_1'}{\partial u \partial v} - \frac{\sigma_1}{\tau_1}\left(a + \frac{\partial \log \theta_1}{\partial v}\right)\frac{\partial \theta_1'}{\partial u} - \frac{\tau_1}{\sigma_1}\left(b + \frac{\partial \log \theta_1}{\partial u}\right)\frac{\partial \theta_1'}{\partial v} = 0. \tag{10}$$

Since the relation between S and S_1 is reciprocal, there exist functions ϕ_1^{-1} and θ_1^{-1} by means of which S is the transform of S_1. We shall show that

$$\theta_1^{-1} = 1, \qquad \phi_1^{-1} = \frac{\phi_1 \theta_1}{\sigma_1 \tau_1}. \tag{11}$$

We remark that as the congruence G_1 is the same, on the assumption that $\theta_1^{-1} = 1$, we must have analogously to (3)

$$\begin{aligned} \frac{\partial}{\partial u}(\rho x_1') &= \phi_1^{-1}\left[\frac{\partial x_1}{\partial u} - \frac{\tau_1}{\sigma_1}\left(b + \frac{\partial \log \theta_1}{\partial u}\right)x_1\right], \\ \frac{\partial}{\partial v}(\rho x_1') &= x_1\left[\frac{\partial \phi_1^{-1}}{\partial v} + \frac{\sigma_1}{\tau_1}\left(a + \frac{\partial \log \theta_1}{\partial v}\right)\phi_1^{-1}\right], \end{aligned} \tag{12}$$

where ρ is to be determined. If the value of x_1' from (5) be substituted in the first of these equations, the result is reducible to the form

$$A\frac{\partial x}{\partial u} + Bx = 0,$$

where A and B are determinate expressions. Since similar equations hold in y, z, and w, A and B must be equal to zero. From these equations we find that ρ is $1/\sigma$ and that ϕ_1^{-1} is of the form (11). It is readily shown that these values satisfy the second of (12).

From (4) and (6) we have

$$\begin{aligned} \frac{\partial^2 \sigma_1}{\partial u \partial v} &= \frac{1}{\theta_1 \phi_1}\frac{\partial \sigma_1}{\partial u}\frac{\partial \sigma_1}{\partial v} + \theta_1 \phi_1 k, \\ \frac{\partial^2 \tau_1}{\partial u \partial v} &= \frac{1}{\theta_1 \phi_1}\frac{\partial \tau_1}{\partial u}\frac{\partial \tau_1}{\partial v} + \theta_1 \phi_1 h, \end{aligned} \tag{13}$$

where h and k are the invariants of (1) and are given by

$$h = \frac{\partial a}{\partial u} + ab - c, \qquad k = \frac{\partial b}{\partial v} + ab - c. \tag{14}$$

By means of (4) and (6) equation (10) may be given the form

$$\frac{\partial^2 \theta_1'}{\partial u \partial v} - \frac{\sigma_1}{\tau_1 \theta_1 \phi_1} \frac{\partial \tau_1}{\partial v} \frac{\partial \theta_1'}{\partial u} - \frac{\tau_1}{\sigma_1 \theta_1 \phi_1} \frac{\partial \sigma_1}{\partial u} \frac{\partial \theta_1'}{\partial v} = 0. \tag{15}$$

In consequence of (13) the adjoint of this equation is reducible to

$$\begin{aligned} &\frac{\partial^2 \phi_1'}{\partial u \partial v} + \frac{\sigma_1}{\tau_1 \theta_1 \phi_1} \frac{\partial \tau_1}{\partial v} \frac{\partial \phi_1'}{\partial u} + \frac{\tau_1}{\sigma_1 \theta_1 \phi_1} \frac{\partial \sigma_1}{\partial u} \frac{\partial \phi_1'}{\partial v} \\ &\qquad + \phi_1' \left[\frac{\sigma_1}{\tau_1} h + \frac{\tau_1}{\sigma_1} k + \frac{1}{\theta_1^2 \phi_1^2} \left(\frac{\partial \sigma_1}{\partial u} \frac{\partial \tau_1}{\partial v} + \frac{\partial \sigma_1}{\partial v} \frac{\partial \tau_1}{\partial u} \right) \right. \\ &\qquad \left. - \frac{1}{\theta_1 \phi_1} \left(\frac{\sigma_1}{\tau_1^2} \frac{\partial \tau_1}{\partial u} \frac{\partial \tau_1}{\partial v} + \frac{\tau_1}{\sigma_1^2} \frac{\partial \sigma_1}{\partial u} \frac{\partial \sigma_1}{\partial v} \right) \right] = 0. \end{aligned} \tag{16}$$

From (11) we find

$$\frac{\partial^2 \phi_1^{-1}}{\partial u \partial v} = \frac{1}{\tau_1^2} \frac{\partial \tau_1}{\partial u} \frac{\partial \tau_1}{\partial v} \frac{\tau_1 + 2\sigma_1}{\tau_1 \theta_1 \phi_1} + \frac{1}{\sigma_1^2} \frac{\partial \sigma_1}{\partial u} \frac{\partial \sigma_1}{\partial v} \frac{\sigma_1 + 2\tau_1}{\sigma_1 \theta_1 \phi_1} - \theta_1 \phi_1 \left(\frac{h}{\tau_1^2} + \frac{k}{\sigma_1^2} \right).$$

Making use of this result, we verify readily that ϕ_1^{-1} is a solution of (16).

If ϕ_1' is any solution of (16), then

$$\frac{\partial^2}{\partial u \partial v} \left(\frac{\phi_1'}{\phi_1^{-1}} \right) = \frac{\tau_1}{\theta_1 \phi_1 \sigma_1} \frac{\partial \sigma_1}{\partial v} \frac{\partial}{\partial u} \left(\frac{\phi_1'}{\phi_1^{-1}} \right) + \frac{\sigma_1}{\theta_1 \phi_1 \tau_1} \frac{\partial \tau_1}{\partial u} \frac{\partial}{\partial v} \left(\frac{\phi_1'}{\phi_1^{-1}} \right).$$

Hence if ϕ_1 and ϕ_2 are two solutions of (2), the equations

$$\frac{\partial}{\partial u} \left(\frac{\phi_{12}}{\phi_1^{-1}} \right) = \sigma_1 \frac{\partial}{\partial u} \left(\frac{\phi_2}{\phi_1} \right), \qquad \frac{\partial}{\partial v} \left(\frac{\phi_{12}}{\phi_1^{-1}} \right) = -\tau_1 \frac{\partial}{\partial v} \left(\frac{\phi_2}{\phi_1} \right) \tag{17}$$

are consistent, and the function ϕ_{12} so defined is a solution of (16).

3. Transformations T in cartesian coördinates. Parallel transformations T

We consider now the case when the point coördinates are non-homogeneous and rectangular. The point equation is of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} + a \frac{\partial \theta}{\partial u} + b \frac{\partial \theta}{\partial v} = 0. \tag{18}$$

When we take $w = 1$, we have from the corresponding equation (9)

$$\frac{\partial w_1}{\partial u} = \tau_1 \frac{\partial}{\partial u} \left(\frac{1}{\theta_1} \right), \qquad \frac{\partial w_1}{\partial v} = -\sigma_1 \frac{\partial}{\partial v} \left(\frac{1}{\theta_1} \right). \tag{19}$$

Moreover, the cartesian coördinates x_1, y_1, z_1 of S_1 are given by equations of the form

$$\frac{\partial}{\partial u} (x_1 w_1) = \tau_1 \frac{\partial}{\partial u} \left(\frac{x}{\theta_1} \right), \qquad \frac{\partial}{\partial v} (x_1 w_1) = -\sigma_1 \frac{\partial}{\partial v} \left(\frac{x}{\theta_1} \right). \tag{20}$$

By means of (19) these equations are reducible to

$$(21)\qquad \begin{aligned} \frac{\partial x_1}{\partial u} &= \frac{\tau_1}{w_1\theta_1^2}\left(\theta_1\frac{\partial x}{\partial u} + (x_1 - x)\frac{\partial \theta_1}{\partial u}\right), \\ \frac{\partial x_1}{\partial v} &= -\frac{\sigma_1}{w_1\theta_1^2}\left(\theta_1\frac{\partial x}{\partial v} + (x_1 - x)\frac{\partial \theta_1}{\partial v}\right). \end{aligned}$$

Now the point equation of S_1 is

$$(22)\qquad \begin{aligned} \frac{\partial^2 \theta_1'}{\partial u \partial v} &+ \left[\frac{\partial \log w_1}{\partial v} - \left(a + \frac{\partial \log \theta_1}{\partial v}\right)\frac{\sigma_1}{\tau_1}\right]\frac{\partial \theta_1'}{\partial u} \\ &+ \left[\frac{\partial \log w_1}{\partial u} - \left(b + \frac{\partial \log \theta_1}{\partial u}\right)\frac{\tau_1}{\sigma_1}\right]\frac{\partial \theta_1'}{\partial v} = 0. \end{aligned}$$

The adjoint of this equation is obtained by replacing ϕ_1' in (16) by ϕ_1'/w_1.

We consider the transformations for which ϕ_1 is any solution of (2) and $\theta_1 = 1$. The corresponding functions σ_1 and τ_1 are given by

$$(23)\qquad \begin{aligned} \sigma_1' &= \int b\phi_1\, du + \left(\frac{\partial \phi_1}{\partial v} - a\phi_1\right) dv, \\ \tau_1' &= \int \left(\frac{\partial \phi_1}{\partial u} - b\phi_1\right) du + a\phi_1\, dv. \end{aligned}$$

It is readily shown that these particular values are in the following relations with the functions σ_1 and τ_1 as given by (4) and (6):

$$(24)\qquad \tau_1' = \frac{\tau_1}{\theta_1} - w_1, \qquad \sigma_1' = \frac{\sigma_1}{\theta_1} + w_1.$$

If $S^{(1)}$ denotes the corresponding transform of S, and its cartesian coördinates are denoted by $x^{(1)}, y^{(1)}, z^{(1)}$, we have from (19) and (20)

$$(25)\qquad \frac{\partial x^{(1)}}{\partial u} = \tau_1'\frac{\partial x}{\partial u}, \qquad \frac{\partial x^{(1)}}{\partial v} = -\sigma_1'\frac{\partial x}{\partial v}.$$

From the form of these equations it is evident that the tangent planes to S and $S^{(1)}$ at corresponding points are parallel. Hence, when $\theta_1 = 1$, we have the *parallel* transformations T.

4. Theorem of permutability of transformations T

Suppose that we have two transforms S_1 and S_2 of S determined by the respective sets of functions σ_1, τ_1 and σ_2, τ_2, where σ_2 and τ_2 are given by (4) and (6) when θ_1 and ϕ_1 are replaced by θ_2 and ϕ_2. A solution θ_{12} of equation (22) is given by the quadratures

$$(26)\qquad \frac{\partial}{\partial u}(w_1\theta_{12}) = \tau_1\frac{\partial}{\partial u}\left(\frac{\theta_2}{\theta_1}\right), \qquad \frac{\partial}{\partial v}(w_1\theta_{12}) = -\sigma_1\frac{\partial}{\partial v}\left(\frac{\theta_2}{\theta_1}\right),$$

which are of the same form as (20).

We consider the surface S_{12} obtained from S_1 by the transformation T determined by θ_{12} and ϕ_{12}, as given by (17), where now

$$\phi_1^{-1} = \frac{w_1\,\theta_1\,\phi_1}{\tau_1\,\sigma_1}. \tag{27}$$

The function w_{12} of this transformation is given by

$$\frac{\partial w_{12}}{\partial u} = \tau_{12}\frac{\partial}{\partial u}\left(\frac{1}{\theta_{12}}\right), \qquad \frac{\partial w_{12}}{\partial v} = -\,\sigma_{12}\frac{\partial}{\partial v}\left(\frac{1}{\theta_{12}}\right), \tag{28}$$

where σ_{12} and τ_{12} are defined by equations analogous to (4) and (6), namely

$$\begin{aligned}\sigma_{12} &= \int \phi_{12}\left[\frac{\partial\theta_{12}}{\partial u} + \theta_{12}\left\{\frac{\partial \log w_1}{\partial u} - \left(b + \frac{\partial \log \theta_1}{\partial u}\right)\frac{\tau_1}{\sigma_1}\right\}\right]du \\ &\qquad + \theta_{12}\left[\frac{\partial\phi_{12}}{\partial v} - \phi_{12}\left\{\frac{\partial \log w_1}{\partial v} - \left(a + \frac{\partial \log \theta_1}{\partial v}\right)\frac{\sigma_1}{\tau_1}\right\}\right]dv, \\ \tau_{12} &= \int \theta_{12}\left[\frac{\partial\phi_{12}}{\partial u} - \phi_{12}\left\{\frac{\partial \log w_1}{\partial u} - \left(b + \frac{\partial \log \theta_1}{\partial u}\right)\frac{\tau_1}{\sigma_1}\right\}\right]du \\ &\qquad + \phi_{12}\left[\frac{\partial\theta_{12}}{\partial v} + \theta_{12}\left\{\frac{\partial \log w_1}{\partial v} - \left(a + \frac{\partial \log \theta_1}{\partial v}\right)\frac{\sigma_1}{\tau_1}\right\}\right]dv.\end{aligned} \tag{29}$$

By means of (17), (26), and (27) these expressions are reducible to

$$\begin{aligned}\sigma_{12} &= \int \phi_{12}\left[-\frac{\tau_1}{w_1}\frac{\partial}{\partial u}\left(\frac{\theta_2}{\theta_1}\right) - \left(b + \frac{\partial \log \theta_1}{\partial u}\right)\frac{\theta_{12}\,\tau_1}{\sigma_1}\right]du \\ &\qquad + \theta_{12}\left[\frac{\phi_{12}\,\tau_1}{\sigma_1\,\phi_1}\left(a\phi_1 - \frac{\partial\phi_1}{\partial v}\right) + \frac{\theta_1\,\phi_1\,w_1}{\sigma_1}\frac{\partial}{\partial v}\left(\frac{\phi_2}{\phi_1}\right)\right]dv, \\ \tau_{12} &= \int \theta_{12}\left[\frac{\phi_{12}\,\sigma_1}{\tau_1\,\phi_1}\left(b\phi_1 - \frac{\partial\phi_1}{\partial u}\right) - \frac{\theta_1\,\phi_1\,w_1}{\tau_1}\frac{\partial}{\partial u}\left(\frac{\phi_2}{\phi_1}\right)\right]du \\ &\qquad + \phi_{12}\left[\frac{\sigma_1}{w_1}\frac{\partial}{\partial v}\left(\frac{\theta_2}{\theta_1}\right) - \left(a + \frac{\partial \log \theta_1}{\partial v}\right)\frac{\theta_{12}\,\sigma_1}{\tau_1}\right]dv.\end{aligned} \tag{30}$$

The coördinates x_{12}, y_{12}, z_{12} of S_{12} are given by equations similar to (21), which are reducible by (26) to

$$\begin{aligned}\frac{\partial x_{12}}{\partial u} &= \frac{\tau_1\,\tau_{12}}{\theta_1^2\,\theta_{12}^2\,w_1\,w_{12}}\left[x_{12}\left\{\theta_{12}\frac{\partial\theta_1}{\partial u} + \theta_1^2\frac{\partial}{\partial u}\left(\frac{\theta_2}{\theta_1}\right)\right\}\right. \\ &\qquad \left. - x_1\,\theta_1^2\frac{\partial}{\partial u}\left(\frac{\theta_2}{\theta_1}\right) + \theta_{12}\,\theta_1^2\frac{\partial}{\partial u}\left(\frac{x}{\theta_1}\right)\right], \\ \frac{\partial x_{12}}{\partial v} &= \frac{\sigma_1\,\sigma_{12}}{\theta_1^2\,\theta_{12}^2\,w_1\,w_{12}}\left[x_{12}\left\{\theta_{12}\frac{\partial\theta_1}{\partial v} + \theta_1^2\frac{\partial}{\partial v}\left(\frac{\theta_2}{\theta_1}\right)\right\}\right. \\ &\qquad \left. - x_1\,\theta_1^2\frac{\partial}{\partial v}\left(\frac{\theta_2}{\theta_1}\right) + \theta_{12}\,\theta_1^2\frac{\partial}{\partial v}\left(\frac{x}{\theta_1}\right)\right].\end{aligned} \tag{31}$$

Proceeding in like manner with S_2, we obtain a transform S_{21} by means of functions θ_{21}, ϕ_{21}, σ_{21}, τ_{21}. The equations similar to (31) are obtained by interchanging the subscripts 1 and 2 in (31). From these two sets of equations it follows that S_{12} and S_{21} are the same surface, if

$$\begin{aligned} &\tau_{12}\,\tau_1\,\theta_2\,\theta_{21}\,w_2\,w_{21} = \tau_{21}\,\tau_2\,\theta_1\,\theta_{12}\,w_1\,w_{12}, \\ &\sigma_{12}\,\sigma_1\,\theta_2\,\theta_{21}\,w_2\,w_{21} = \sigma_{21}\,\sigma_2\,\theta_1\,\theta_{12}\,w_1\,w_{12}, \end{aligned} \tag{32}$$

and

$$x_{12} = \Theta_{12}\,(x_1\,\theta_2\,\theta_{21} + x_2\,\theta_1\,\theta_{12} - x\,\theta_{12}\,\theta_{21}), \tag{33}$$

where

$$\frac{1}{\Theta_{12}} = \theta_2\,\theta_{21} + \theta_1\,\theta_{12} - \theta_{12}\,\theta_{21}. \tag{34}$$

When we express the condition that this value of x_{12} shall satisfy equations (31), we get

$$\begin{aligned} &\left[\tau_{12}\,\tau_1 - \theta_1\,\theta_{12}\,w_1\,w_{12}\,\Theta_{12}\left(\frac{\theta_2\,\theta_{21}\,\tau_1}{\theta_1\,w_1} + \frac{\theta_1\,\theta_{12}\,\tau_2}{\theta_2\,w_2} - \theta_{12}\,\theta_{21}\right)\right]\left[x_1\left(-\theta_{21}\frac{\partial\theta_2}{\partial u}\right.\right. \\ &\qquad\left. - \theta_2\frac{\partial\theta_1}{\partial u} + \theta_1\frac{\partial\theta_2}{\partial u}\right) + x_2\left(\theta_{12}\frac{\partial\theta_1}{\partial u} + \theta_2\frac{\partial\theta_1}{\partial u} - \theta_1\frac{\partial\theta_2}{\partial u}\right) \\ &\qquad\left. - x\left(\theta_{21}\frac{\partial\theta_2}{\partial u} + \theta_{12}\frac{\partial\theta_1}{\partial u}\right) + \frac{1}{\Theta_{12}}\frac{\partial x}{\partial u}\right] = 0, \\ &\left[\sigma_{12}\,\sigma_1 + \theta_1\,\theta_{12}\,w_1\,w_{12}\,\Theta_{12}\left(\frac{\theta_2\,\theta_{21}\,\sigma_1}{\theta_1\,w_1} + \frac{\theta_1\,\theta_{12}\,\sigma_2}{\theta_2\,w_2} + \theta_{12}\,\theta_{21}\right)\right]\left[x_1\left(-\theta_{21}\frac{\partial\theta_2}{\partial v}\right.\right. \\ &\qquad\left. - \theta_2\frac{\partial\theta_1}{\partial v} + \theta_1\frac{\partial\theta_2}{\partial v}\right) + x_2\left(\theta_{12}\frac{\partial\theta_1}{\partial v} + \theta_2\frac{\partial\theta_1}{\partial v} - \theta_1\frac{\partial\theta_2}{\partial v}\right) \\ &\qquad\left. - x\left(\theta_{21}\frac{\partial\theta_2}{\partial v} + \theta_{12}\frac{\partial\theta_1}{\partial v}\right) + \frac{1}{\Theta_{12}}\frac{\partial x}{\partial v}\right] = 0. \end{aligned}$$

On the assumption that θ_1 and θ_2 are independent, these are equivalent to

$$\begin{aligned} \sigma_1\,\sigma_{12} &= -\,\theta_1\,\theta_{12}\,w_1\,w_{12}\,\Theta_{12}\left(\frac{\theta_2\,\theta_{21}\,\sigma_1}{\theta_1\,w_1} + \frac{\theta_1\,\theta_{12}\,\sigma_2}{\theta_2\,w_2} + \theta_{12}\,\theta_{21}\right), \\ \tau_1\,\tau_{12} &= \theta_1\,\theta_{12}\,w_1\,w_{12}\,\Theta_{12}\left(\frac{\theta_2\,\theta_{21}\,\tau_1}{\theta_1\,w_1} + \frac{\theta_1\,\theta_{12}\,\tau_2}{\theta_2\,w_2} - \theta_{12}\,\theta_{21}\right). \end{aligned}$$

From these equations and one analogous to (8) we obtain

$$\phi_{12} = \frac{\theta_1^2\,\theta_{12}\,w_1\,w_{12}}{\sigma_1\,\tau_1}\Theta_{12}\left(-\,\phi_1\,\theta_{21} + (\sigma_1\,\tau_2 - \sigma_2\,\tau_1)\frac{1}{\theta_2\,w_2}\right).$$

When this value is substituted in (17), we find that the transformation func-

tions must have the expressions

$$
(35)\qquad
\begin{aligned}
w_{12} &= \frac{w_2}{\theta_1\,\theta_{12}\,\Theta_{12}},\\
\phi_{12} &= \frac{w_1\,\theta_1\,\phi_1}{\tau_1\,\sigma_1}\left(-\,w_2\,\theta_{21} + \frac{1}{\theta_2\,\phi_1}(\tau_2\,\sigma_1 - \tau_1\,\sigma_2)\right),\\
\sigma_{12} &= -\,\frac{w_1\,w_2}{\sigma_1}\left(\frac{\theta_2\,\theta_{21}\,\sigma_1}{\theta_1\,w_1} + \frac{\theta_1\,\theta_{12}\,\sigma_2}{\theta_2\,w_2} + \theta_{12}\,\theta_{21}\right),\\
\tau_{12} &= \frac{w_1\,w_2}{\tau_1}\left(\frac{\theta_2\,\theta_{21}\,\tau_1}{\theta_1\,w_1} + \frac{\theta_1\,\theta_{12}\,\tau_2}{\theta_2\,w_2} - \theta_{12}\,\theta_{21}\right).
\end{aligned}
$$

It is readily verified that these values for σ_{12} and τ_{12} satisfy equations (30).

In consequence of (35) equation (33) can be written

$$
(36)\qquad \theta_1\,\theta_{12}\,w_{12}\,x_{12} = w_2(\theta_2\,\theta_{21}\,x_1 + \theta_1\,\theta_{12}\,x_2 - \theta_{12}\,\theta_{21}\,x).
$$

Thus we have established a theorem of permutability of general transformations T. There are two arbitrary constants involved, namely in the determination of θ_{12} by (26) and of θ_{21} by

$$
(37)\qquad \frac{\partial}{\partial u}(w_2\,\theta_{21}) = \tau_2\frac{\partial}{\partial u}\left(\frac{\theta_1}{\theta_2}\right),\qquad \frac{\partial}{\partial v}(w_2\,\theta_{21}) = -\,\sigma_2\frac{\partial}{\partial v}\left(\frac{\theta_1}{\theta_2}\right).
$$

Accordingly we formulate

THEOREM 1. *If S_1 and S_2 are two transforms of S, there exist ∞^2 surfaces S_{12}, each of which is a transform of both S_1 and S_2; and their complete determination requires two quadratures.*

We say that four such surfaces S, S_1, S_2, S_{12} form a *quatern*.

We consider, in particular, the case where S_2 is parallel to S. If we take $\theta_2 = 1$, in accordance with (19) and (26) we have $\theta_{12} = 1$ as one solution. Now (33) becomes

$$
(38)\qquad (x_{12} - x_2)\,\theta_1 = (x_1 - x)\,\theta_{21}.
$$

Hence we have

THEOREM 2. *If S_2 is parallel to S and S_1 is any transform of S, one of the surfaces S_{12} is parallel to S_1; moreover the lines joining corresponding points on S_{12} and S_2 and on S and S_1 are parallel.*

If both S_1 and S_2 are parallel to S, the functions θ_1 and θ_2 are constants. Hence from (33) it follows that x_{12} is a linear function of x, x_1, x_2, with constant coefficients, and consequently S_{12} also is parallel to S.

5. Envelope of the planes of a quatern

If M, M_1, M_2, M_{12} are corresponding points of four surfaces of a quatern, it follows from (35) and (36) that these four points lie in a plane π. Since

this plane contains the lines MM_1 and MM_2 which generate congruences conjugate to the parametric conjugate system on S, it envelopes a surface Σ upon which the parametric curves form a conjugate system, as follows from the general theory of congruences.* Moreover, if Π is the point of the envelope corresponding to M on S, the tangent at Π to one of these curves passes through the focal points F_1' and F_2' of the lines MM_1 and MM_2 respectively, and the tangent to the other curve passes through the focal points F_1'' and F_2''. We will now find the coördinates of Π.

In cartesian coördinates equations (5) and (7) are of the form

$$w_1 x_1 = x_1' \sigma_1' - \frac{\sigma_1}{\theta_1} x, \qquad w_1 x_1 = - x_1'' \tau_1' + \frac{\tau_1}{\theta_1} x, \tag{39}$$

where now $x_1' \sigma_1'$ and $x_1'' \tau_1'$ are respectively equal to the right-hand members of (3). Similar equations with subscripts 2 hold for the congruence of lines MM_2.

The cartesian coördinates ξ, η, ζ of Π are given by equations of the form

$$\begin{aligned} \xi &= \frac{1}{\sigma_1'}\left(x_1 w_1 + \frac{\sigma_1}{\theta_1} x\right) + t_1\left[\frac{1}{\sigma_1'}\left(x_1 w_1 + \frac{\sigma_1}{\theta_1} x\right) - \frac{1}{\sigma_2'}\left(x_2 w_2 + \frac{\sigma_2}{\theta_2} x\right)\right], \\ &= \frac{1}{\tau_1'}\left(- x_1 w_1 + \frac{\tau_1}{\theta_1} x\right) + t_2\left[\frac{1}{\tau_1'}\left(- x_1 w_1 + \frac{\tau_1}{\theta_1} x\right)\right. \\ &\qquad \left. - \frac{1}{\tau_2'}\left(- x_2 w_2 + \frac{\tau_2}{\theta_2} x\right)\right], \end{aligned} \tag{40}$$

where t_1 and t_2 are to be determined. When these two expressions for ξ are equated, we get an equation of the form

$$Ax + Bx_1 + Cx_2 = 0,$$

where A, B, and C are determinate functions. Since similar equations in the y's and z's also must hold, we must have $A = B = C = 0$. From the first two of these equations we get

$$\phi_1\left(w_2 + \frac{\sigma_2}{\theta_2}\right) + t_1\left(w_2 \phi_1 - w_1 \phi_2 + \frac{\sigma_2 \tau_1 - \sigma_1 \tau_2}{\theta_1 \theta_2}\right) = 0,$$

$$\phi_1\left(w_2 - \frac{\tau_2}{\theta_2}\right) + t_2\left(w_2 \phi_1 - w_1 \phi_2 + \frac{\sigma_2 \tau_1 - \sigma_1 \tau_2}{\theta_1 \theta_2}\right) = 0.$$

These values satisfy $C = 0$, and when substituted in the above expressions

*Guichard, Annales de l'école normale superieure, ser. 3, vol. 14 (1897).

for ξ we get

$$\xi\left(w_2\,\phi_1 - w_1\,\phi_2 + \frac{\sigma_2\,\tau_1 - \sigma_1\,\tau_2}{\theta_1\,\theta_2}\right) = w_2\,\phi_1\,x_2 - w_1\,\phi_2\,x_1 - \frac{\sigma_1\,\tau_2 - \sigma_2\,\tau_1}{\theta_1\,\theta_2}x. \tag{41}$$

We shall find the functions of the theorem of permutability when homogeneous coördinates are used. Now the functions θ_{12} and θ_{21} are given by

$$\frac{\partial\theta_{ij}}{\partial u} = \tau_i\frac{\partial}{\partial u}\left(\frac{\theta_j}{\theta_i}\right), \qquad \frac{\partial\theta_{ij}}{\partial v} = -\,\sigma_i\frac{\partial}{\partial v}\left(\frac{\theta_j}{\theta_i}\right) \qquad \left(\begin{matrix} i=1,2, \\ j=1,2, \end{matrix}\quad i \neq j\right), \tag{42}$$

and the coördinates $x_{12}, \cdots, w_{12}$ of S_{12} must satisfy the equations of the form

$$\frac{\partial x_{ij}}{\partial u} = \tau_{ij}\frac{\partial}{\partial u}\left(\frac{x_i}{\theta_{ij}}\right), \qquad \frac{\partial x_{ij}}{\partial v} = -\,\sigma_{ij}\frac{\partial}{\partial v}\left(\frac{x_i}{\theta_{ij}}\right) \qquad \left(\begin{matrix} i=1,2, \\ j=1,2, \end{matrix}\quad i \neq j\right). \tag{43}$$

From (35) it follows that the functions τ_{12}, σ_{12}, ϕ_{12} are of the form

$$\begin{gathered}
\tau_1\,\tau_{12} = \tau_2\,\tau_{21} = \frac{\theta_2\,\theta_{21}\,\tau_1}{\theta_1} + \frac{\theta_1\,\theta_{12}\,\tau_2}{\theta_2} - \theta_{12}\,\theta_{21}, \\
\sigma_1\,\sigma_{12} = \sigma_2\,\sigma_{21} = -\left(\frac{\theta_2\,\theta_{21}\,\sigma_1}{\theta_1} + \frac{\theta_1\,\theta_{12}\,\sigma_2}{\theta_2} + \theta_{12}\,\theta_{21}\right), \\
\phi_{12} = \frac{\theta_1\,\phi_1}{\sigma_1\,\tau_1}\left(-\,\theta_{21} + (\sigma_1\,\tau_2 - \sigma_2\,\tau_1)\frac{1}{\theta_2\,\phi_1}\right).
\end{gathered} \tag{44}$$

Moreover, the coördinate x_{12} is expressed by

$$\theta_1\,\theta_{12}\,x_{12} = \theta_2\,\theta_{21}\,x_1 + \theta_1\,\theta_{12}\,x_2 - \theta_{12}\,\theta_{21}\,x. \tag{45}$$

6. Transformations T determined by the same function ϕ

We consider now the relation of two transformations determined by θ_1 and θ_2 respectively but by the same function ϕ_1. If we put

$$\begin{aligned}
(\tau_1)_2 &= \int \theta_2\left(\frac{\partial\phi_1}{\partial u} - b\phi_1\right)du + \phi_1\left(\frac{\partial\theta_2}{\partial v} + a\theta_2\right)dv, \\
(\sigma_1)_2 &= \int \phi_1\left(\frac{\partial\theta_2}{\partial u} + b\theta_2\right)du + \theta_2\left(\frac{\partial\phi_1}{\partial v} - a\phi_1\right)dv,
\end{aligned}$$

we have in consequence of (26)

$$(\tau_1)_2 = \frac{\theta_2}{\theta_1}\tau_1 - w_1\,\theta_{12}, \qquad (\sigma_1)_2 = \frac{\theta_2}{\theta_1}\sigma_1 + w_1\,\theta_{12}. \tag{46}$$

When these values are substituted in equations analogous to (20), namely

$$\frac{\partial}{\partial u}(x_2\,w_2) = (\tau_1)_2\frac{\partial}{\partial u}\left(\frac{x}{\theta_2}\right), \qquad \frac{\partial}{\partial v}(x_2\,w_2) = -\,(\sigma_1)_2\frac{\partial}{\partial v}\left(\frac{x}{\theta_2}\right),$$

the latter can be integrated in the form

$$x_2 w_2 = x_1 w_1 - x w_1 \frac{\theta_{12}}{\theta_2}. \tag{47}$$

In like manner from equations analogous to (19) we get

$$w_2 = \frac{w_1}{\theta_2}(\theta_2 - \theta_{12}) + c, \tag{48}$$

where c is a constant. By means of this result equation (47) can be given the form

$$(x_2 - x) w_2 = (x_1 - x) w_1 - cx. \tag{49}$$

From this it is seen that the congruence G_2 of lines joining corresponding points on S and S_2 is the same as the congruence G_1 only in case $c = 0$ in (48).

When the above expressions for $(\sigma_1)_2$ and $(\tau_1)_2$ are substituted in (37), we find

$$w_2 \theta_{21} = - w_1 \theta_{12} \frac{\theta_1}{\theta_2} + k, \tag{50}$$

where k is an additive constant.

From (33) and (34) we obtain for the present case

$$\begin{gathered} \frac{w_2}{\Theta_{12}} = (\theta_2 - \theta_{12}) k + \theta_1 \theta_{12} c, \qquad w_{12} = \frac{(\theta_2 - \theta_{12}) k}{\theta_1 \theta_{12}} + c, \\ x_{12} (k(\theta_2 - \theta_{12}) + c\theta_1 \theta_{12}) = k \frac{w_2 \theta_2}{w_1} x_2. \end{gathered} \tag{51}$$

Hence if $k = 0$, the surface S_{12} reduces to a point; if $c = 0$, it coincides with S_2.

In the inverse transformation from S_1 to S the function w_1^{-1} has the value $1/\theta_1$, as is evident from (20). If we look upon S and S_{12} as transforms of S_1, the analogue of equation (47) is

$$x_{12} w_{12} = w_1^{-1} \left(x - x_1 \frac{\theta_2}{\theta_{12}} \right). \tag{52}$$

This equation is satisfied by the value of x_{12}, given by (51), provided $k = -1$.

Incidentally we remark that the last of (51) can be written

$$x_{12} = x_2 \bigg/ \left(1 - \frac{c}{k} \theta_{21} \right). \tag{53}$$

The denominator of this equation is a solution of the point equation of S_2. Moreover, corresponding points on S_2 and S_{12} are on a line through the origin. This is a type of transformations which we will consider later (§ 13); we call them *radial* transformations.

Accordingly we have

THEOREM 3. *When a transform S_1 of S is known, the determination of another transform S_2 with the same function ϕ requires a single quadrature; then the fourth surface of the quatern is a radial transform of S_2.*

7. ANOTHER FORM OF TRANSFORMATIONS T

Particular importance attaches to the results of the preceding section when we take a parallel surface for S_2. As in § 3, we call it $S^{(1)}$ and its coördinates $x^{(1)}, y^{(1)}, z^{(1)}$. We take $\theta_2 = 1$, then $\theta_{12} = 1$. Also we take $c = 1$. Then (47) assumes the desired form

$$x_1 = x + \frac{x^{(1)}}{w_1}. \tag{54}$$

In consequence of (24) equations (19) can be written

$$\frac{\partial}{\partial u}(w_1 \theta_1) = -\tau_1' \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial}{\partial v}(w_1 \theta_1) = \sigma_1' \frac{\partial \theta_1}{\partial v}.$$

Hence if we put

$$w_1 \theta_1 = -\theta_1^{(1)}, \tag{55}$$

we have

$$\frac{\partial \theta_1^{(1)}}{\partial u} = \tau_1' \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1^{(1)}}{\partial v} = -\sigma_1' \frac{\partial \theta_1}{\partial v}, \tag{56}$$

which are similar to (25). By means of (55) equation (54) is reducible to*

$$x_1 = x - \frac{\theta_1}{\theta_1^{(1)}} x^{(1)}. \tag{57}$$

The significance of this result is that the problem of finding transformations T is reduced to that of finding parallel transforms and the integration of equation (1).

Equations (57) enable us to show that when we take

$$\theta_1 = ax + by + cz, \qquad \theta_1^{(1)} = ax^{(1)} + by^{(1)} + cz^{(1)},$$

where a, b, c are constants, then S_1 is the plane $ax_1 + by_1 + cz_1 = 0$.

In consequence of (24) and (54) equations (39) giving the coördinates of the focal points of the congruence G_1 are reducible to

$$x_1' = x + \frac{x^{(1)}}{\sigma_1'}, \qquad x_1'' = x - \frac{x^{(1)}}{\tau_1'}. \tag{58}$$

As an application of these results we seek the condition that S_1 shall be normal to the lines of the congruence G_1. From (57) it is seen that $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ are the direction-parameters of the lines of this congruence. Hence

* Cf. Jonas, l. c., p. 102.

we must have

$$\sum x^{(1)} \frac{\partial x_1}{\partial u} = 0, \qquad \sum x^{(1)} \frac{\partial x_1}{\partial v} = 0.$$

Substituting the values of x_1, y_1, z_1 from (57), we have to within a constant factor

$$\theta_1^{(1)} = \sqrt{x^{(1)2} + y^{(1)2} + z^{(1)2}}. \tag{59}$$

From (22) it follows that the point equation of $S^{(1)}$ is

$$\frac{\partial^2 \theta_1^{(1)}}{\partial u \partial v} - a \frac{\sigma_1'}{\tau_1'} \frac{\partial \theta_1^{(1)}}{\partial u} - b \frac{\tau_1'}{\sigma_1'} \frac{\partial \theta_1^{(1)}}{\partial v} = 0.$$

Expressing the condition that the above value of $\theta_1^{(1)}$ satisfies this equation, we get

$$\sum \frac{\partial x^{(1)}}{\partial u} \frac{\partial x^{(1)}}{\partial v} - \frac{\partial \theta_1^{(1)}}{\partial u} \frac{\partial \theta_1^{(1)}}{\partial v} = 0,$$

and consequently

$$\sum \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial \theta_1}{\partial u} \frac{\partial \theta_1}{\partial v} = 0.$$

But this is the condition that $x^2 + y^2 + z^2 - \theta_1^2$ also is a solution of the point equation of S. Moreover, it follows from (59) and (57) that in this case θ_1 is the distance from S to S_1. Hence from this point of view we have established the known

THEOREM 4. *When the developables of the congruence of normals to a surface S_1 meet a surface S in a conjugate system, the function t giving the distance between corresponding points on S and S_1 is a solution of the point equation of S as is also the function $x^2 + y^2 + z^2 - t^2$.*

Returning to the general case, we have from (50) in consequence of (55)

$$\theta_{21} = \theta_1^{(1)} - 1. \tag{60}$$

We have taken $k = -1$ so that (52) shall hold. Then S_{12} is the parallel $S_1^{(1)}$ of S_1 by means of which S is obtained from S_1. Consequently the present form of (53) is

$$x_1^{(1)} = \frac{x^{(1)}}{\theta_1^{(1)}}. \tag{61}$$

8. Another form of the equations of the theorem of permutability

Suppose now that we have two transforms S_1 and S_2 of S; we wish to give the theorem of permutability a new form in view of the preceding results. Evidently the functions θ_{12} and θ_{21} are given by expressions analogous to (57), namely

$$\theta_{12} = \theta_2 - \frac{\theta_1}{\theta_1^{(1)}} \theta_2^{(1)} \qquad \theta_{21} = \theta_1 - \frac{\theta_2}{\theta_2^{(2)}} \theta_1^{(2)}, \tag{62}$$

where $\theta_i^{(j)}$ is defined by

$$\frac{\partial \theta_i^{(j)}}{\partial u} = \tau_j' \frac{\partial \theta_i}{\partial u}, \qquad \frac{\partial \theta_i^{(j)}}{\partial v} = -\sigma_j' \frac{\partial \theta_i}{\partial v}, \tag{63}$$

σ_j' and τ_j' being given by equations obtained from (23) on replacing 1 by j. Now

$$\frac{1}{\Theta_{12}} = \frac{\theta_1 \theta_2}{\theta_1^{(1)} \theta_2^{(2)}} (\theta_1^{(1)} \theta_2^{(2)} - \theta_2^{(1)} \theta_1^{(2)}),$$

and from (33) we have

$$\begin{aligned} (\theta_1^{(1)} \theta_2^{(2)} - \theta_2^{(1)} \theta_1^{(2)}) (x_{12} - x) \\ = (\theta_1^{(2)} \theta_2 - \theta_2^{(2)} \theta_1) x^{(1)} + (\theta_2^{(1)} \theta_1 - \theta_1^{(1)} \theta_2) x^{(2)}. \end{aligned} \tag{64}$$

From this equation and (57) we obtain

$$(\theta_1^{(1)} \theta_2^{(2)} - \theta_2^{(1)} \theta_1^{(2)}) (x_{12} - x_1) = (\theta_1^{(1)} \theta_2 - \theta_2^{(1)} \theta_1) \left(\frac{\theta_1^{(2)}}{\theta_1^{(1)}} x^{(1)} - x^{(2)} \right). \tag{65}$$

We note that the expression in the last parenthesis is similar in form to the right-hand member of (57). Hence if we put

$$x_1^{(2)} = x^{(2)} - \frac{\theta_1^{(2)}}{\theta_1^{(1)}} x^{(1)}, \tag{66}$$

the surface $S_1^{(2)}$ whose coördinates are $x_1^{(2)}$, $y_1^{(2)}$, $z_1^{(2)}$ is a transform of $S^{(2)}$.

If equation (66) be differentiated, the resulting equations are reducible to

$$\begin{aligned} \frac{\partial x_1^{(2)}}{\partial u} &= \left(\tau_2' - \frac{\theta_1^{(2)}}{\theta_1^{(1)}} \tau_1' \right) \frac{\theta_1 w_1}{\tau_1} \frac{\partial x_1}{\partial u}, \\ \frac{\partial x_1^{(2)}}{\partial v} &= \left(\sigma_2' - \frac{\theta_1^{(2)}}{\theta_1^{(1)}} \sigma_1' \right) \frac{\theta_1 w_1}{\sigma_1} \frac{\partial x_1}{\partial v}. \end{aligned} \tag{67}$$

Hence $S_1^{(2)}$ is parallel to S_1. We wish to show that it is the parallel surface whose coördinates enable the equations of the transformation from S_1 to S_{12} to be given a form similar to (57). The first derivatives of the coördinates of this desired parallel surface are equal to

$$\tau_{12}' \frac{\partial x_1}{\partial u}, \qquad -\sigma_{12}' \frac{\partial x_1}{\partial v},$$

where in consequence of (24), (35), (57), and (55)

$$\begin{aligned} \tau_{12}' &= \frac{\tau_{12}}{\theta_{12}} - w_{12} = \frac{w_1 \theta_1}{\tau_1} \left(\tau_2' - \frac{\theta_1^{(2)}}{\theta_1^{(1)}} \tau_1' \right), \\ \sigma_{12}' &= \frac{\sigma_{12}}{\theta_{12}} + w_{12} = -\frac{w_1 \theta_1}{\sigma_1} \left(\sigma_2' - \frac{\theta_1^{(2)}}{\theta_1^{(1)}} \sigma_1' \right). \end{aligned} \tag{68}$$

Comparing these results with (67) we find that $S_1^{(2)}$ is the desired surface.

If we write (65) in the form

$$x_{12} = x_1 - \frac{\theta_2 - \frac{\theta_1}{\theta_1^{(1)}}\theta_2^{(1)}}{\theta_2^{(2)} - \frac{\theta_1^{(2)}}{\theta_1^{(1)}}\theta_2^{(1)}} x_1^{(2)}, \tag{69}$$

we note its similarity to (57).

9. Transformations K

We consider now the particular conjugate systems for which the invariants h and k of the point equation are equal. From (14) it is seen that in this case the point equation may be written

$$\frac{\partial^2 \theta}{\partial u \partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial \theta}{\partial v} = 0. \tag{70}$$

From (22) it is seen that the parametric system on S_1 will have equal invariants, if σ_1 and τ_1 are equal. From (4) and (6) it follows that to within a constant factor we must have

$$\phi_1 = 2\theta_1 \rho. \tag{71}$$

It is readily verified that this value of ϕ_1 satisfies the equation adjoint to (70). From (8) we have

$$\sigma_1 = \tau_1 = \theta_1^2 \rho. \tag{72}$$

Now equations (21) become

$$\begin{aligned} \frac{\partial x_1}{\partial u} &= \frac{\rho}{w_1}\left(\theta_1 \frac{\partial x}{\partial u} + (x_1 - x)\frac{\partial \theta_1}{\partial u}\right), \\ \frac{\partial x_1}{\partial v} &= -\frac{\rho}{w_1}\left(\theta_1 \frac{\partial x}{\partial v} + (x_1 - x)\frac{\partial \theta_1}{\partial v}\right), \end{aligned} \tag{73}$$

where w_1 is given by

$$\frac{\partial w_1}{\partial u} = -\rho\frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial w_1}{\partial v} = \rho\frac{\partial \theta_1}{\partial v}. \tag{74}$$

This transformation is the same which we have considered at length in a former paper, and called a transformation K.*

Equations (24) reduce to

$$\sigma_1' = \rho\theta_1 + w_1, \qquad \tau_1' = \rho\theta_1 - w_1,$$

and consequently the expressions (39) for the coördinates of the focal points

* M_1, p. 400.

of the congruence G_1 become

$$x_1' = \frac{\rho\theta_1 x + w_1 x_1}{\rho\theta_1 + w_1}, \qquad x_1'' = \frac{\rho\theta_1 x - w_1 x_1}{\rho\theta_1 - w_1}.$$

The foregoing results are stated in

THEOREM 5. *When a surface S is referred to a conjugate system with equal point invariants and θ_1 is any solution of the point equation of S, the surface S_1 whose coördinates are given by quadratures of the form (73), is referred to a conjugate system with equal point invariants, and the developables of the lines jo ning corresponding points on S and S_1 meet these surfaces in these parametric curves. Moreover, the focal points of the congruence are harmonic to the corresponding points on S and S_1.*

We assume that S_1 and S_2 are two surfaces in the relation of transformations K with S, and apply the theorem of permutability. Equations (26) and (37) are now reducible to

$$\begin{aligned} \frac{\partial}{\partial u}(w_i\theta_{ij}) &= \rho\left(\theta_i\frac{\partial\theta_j}{\partial u} - \theta_j\frac{\partial\theta_i}{\partial u}\right), \\ \frac{\partial}{\partial v}(w_i\theta_{ij}) &= -\rho\left(\theta_i\frac{\partial\theta_j}{\partial v} - \theta_j\frac{\partial\theta_i}{\partial v}\right) \end{aligned} \qquad \left(\begin{matrix} i = 1, 2, \\ j = 1, 2, \end{matrix} \quad i \neq j\right), \tag{75}$$

from which it follows that

$$w_2\theta_{21} + w_1\theta_{12} = \text{const.}$$

From (35) it is seen that the necessary and sufficient condition that $\sigma_{12} = \tau_{12}$ is

$$w_2\theta_{21} + w_1\theta_{12} = 0. \tag{76}$$

Hence we have

THEOREM 6. *When S_1 and S_2 are two surfaces in the relation of transformations K with a surface S, of the ∞^2 surfaces S_{12} forming quaterns with them in accordance with the theorem of permutability of transformations T, ∞^1 are in the relation of transformations K with S_1 and S_2.*

From (35) we have

$$\theta_1 w_{12} = w_1(\theta_{12} - \theta_2) + w_2\theta_1, \qquad \theta_2 w_{21} = w_2(\theta_{21} - \theta_1) + w_1\theta_2, \tag{77}$$

and from (36)

$$\theta_1 w_{12} x_{12} = -w_1\theta_2 x_1 + w_2\theta_1 x_2 + w_1\theta_{12} x. \tag{78}$$

The coördinates ξ, η, ζ, of Π, the point of contact of the plane π with its envelope, as given by (41), are expressible in the form

$$\xi = -\frac{w_2\theta_1 x_2 - w_1\theta_2 x_1}{w_2\theta_1 - w_1\theta_2} = -\frac{\theta_1 w_{12} x_{12} - w_1\theta_{12} x}{\theta_1 w_{12} - w_1\theta_{12}}. \tag{79}$$

Hence Π is the intersection of the lines MM_{12} and $M_1 M_2$; consequently the points M_{12} of the ∞^1 surfaces S_{12} lie on the line MM_{12}. Therefore we have

THEOREM 7. *If S, S_1, S_2, S_{12} are four surfaces of a quatern for transformations K, the plane π of the four corresponding points M, M_1, M_2, M_{12} touches its envelope in the intersection Π of the lines MM_{12} and $M_1 M_2$; the parametric lines on the envelope form a conjugate system whose tangents are harmonic to the lines MM_{12} and $M_1 M_2$, and contain the focal points of the lines MM_1, MM_2, $M_{12} M_1$, $M_{12} M_2$ for the congruences generated by them.**

10. TRANSFORMATIONS T IN TANGENTIAL COÖRDINATES

When a surface S is referred to a conjugate system, if x, y, z, w and X, Y, Z, W are the point and tangential coördinates respectively of S so that

$$Xx + Yy + Zz + Ww = 0, \tag{80}$$

then X, Y, Z, W satisfy an equation of the form

$$\frac{\partial^2 \lambda}{\partial u \partial v} + \alpha \frac{\partial \lambda}{\partial u} + \beta \frac{\partial \lambda}{\partial v} + \gamma \lambda = 0. \tag{81}$$

Evidently the analytical theory of § 1 is independent of the geometrical interpretation there given, and has a meaning when applied to equation (81). This we will give and study the relation between the two sets of equations.

The adjoint of (81) is

$$\frac{\partial^2 \mu}{\partial u \partial v} - \alpha \frac{\partial \mu}{\partial u} - \beta \frac{\partial \mu}{\partial v} + \left(\gamma - \frac{\partial \alpha}{\partial u} - \frac{\partial \beta}{\partial v} \right) \mu = 0. \tag{82}$$

If λ_1 and μ_1 are solutions of these equations, the following integrals have a meaning:

$$\begin{aligned} \bar{\sigma}_1 &= \int \mu_1 \left(\frac{\partial \lambda_1}{\partial u} + \beta \lambda_1 \right) du + \lambda_1 \left(\frac{\partial \mu_1}{\partial v} - \alpha \mu_1 \right) dv, \\ \bar{\tau}_1 &= \int \lambda_1 \left(\frac{\partial \mu_1}{\partial u} - \beta \mu_1 \right) du + \mu_1 \left(\frac{\partial \lambda_1}{\partial v} + \alpha \lambda_1 \right) dv. \end{aligned} \tag{83}$$

The functions X_1, Y_1, Z_1, W_1 defined by equations of the form

$$\frac{\partial X_1}{\partial u} = \bar{\tau}_1 \frac{\partial}{\partial u} \left(\frac{X}{\lambda_1} \right), \qquad \frac{\partial X_1}{\partial v} = -\bar{\sigma}_1 \frac{\partial}{\partial v} \left(\frac{X}{\lambda_1} \right), \tag{84}$$

are the tangential coördinates of a second surface, upon which the parametric curves form a conjugate system.

* Cf. M_1, p. 409.

In addition to (80) we have the equations of condition

$$X\frac{\partial x}{\partial u} + Y\frac{\partial y}{\partial u} + Z\frac{\partial z}{\partial u} + W\frac{\partial w}{\partial u} = 0, \tag{85}$$
$$X\frac{\partial x}{\partial v} + Y\frac{\partial y}{\partial v} + Z\frac{\partial z}{\partial v} + W\frac{\partial w}{\partial v} = 0.$$

Consider the function

$$\theta_1 = X_1' x + Y_1' y + Z_1' z + W_1' w, \tag{86}$$

where X_1', Y_1', Z_1', W_1' are given by integrals of the form

$$X_1' = \int \mu_1\left(\frac{\partial X}{\partial u} + \beta X\right) du + X\left(\frac{\partial \mu_1}{\partial v} - \alpha\mu_1\right) dv. \tag{87}$$

In consequence of (81) and (85) we have

$$\frac{\partial \theta_1}{\partial u} = \sum \frac{\partial x}{\partial u} X_1' + \frac{\partial w}{\partial u} W_1', \qquad \frac{\partial \theta_1}{\partial v} = \sum \frac{\partial x}{\partial v} X_1' + \frac{\partial w}{\partial v} W_1', \tag{88}$$
$$\frac{\partial^2 \theta_1}{\partial u \partial v} = \sum \frac{\partial^2 x}{\partial u \partial v} X_1' + \frac{\partial^2 w}{\partial u \partial v} W_1',$$

where as usual the symbol $\sum$ signifies the sum for three terms in x, y, z. Hence θ_1 is a solution of equation (1).*

In like manner it can be shown that λ_1 given by

$$\lambda_1 = Xx_1' + Yy_1' + Zz_1' + Ww_1', \tag{89}$$

where x_1', y_1', z_1', w_1' are given by equations of the form (3), is a solution of (81). It is our purpose to show that equations (9) and (84) define the same transformation of S, when θ_1 and λ_1 are given by (86) and (89).

The analogue of equation (5) is

$$X_1 = X_1' - \frac{\bar{\sigma}_1}{\lambda_1} X. \tag{90}$$

From (9) it follows that the points T_1 and T_2, whose coördinates ξ_1, η_1, ζ_1, ω_1; ξ_2, η_2, ζ_2, ω_2 are of the form

$$\xi_1 = \frac{\partial}{\partial v}\left(\frac{x}{\theta_1}\right), \qquad \xi_2 = \frac{\partial}{\partial u}\left(\frac{x}{\theta_1}\right), \tag{91}$$

are the intersections of corresponding tangents to the parametric curves on S and S_1. Since

$$\frac{\partial \xi_2}{\partial v} = \frac{\partial \xi_1}{\partial u} = -\left(\frac{\partial \log \theta_1}{\partial v} + a\right)\xi_1 - \left(\frac{\partial \log \theta_1}{\partial u} + b\right)\xi_2, \tag{92}$$

* Cf. Darboux, *Leçons*, vol. 2, p. 188.

the points T_1 and T_2 are the focal points of the congruence of lines $T_1 T_2$. These lines are the intersections of the tangent planes to S and S_1.

We shall show that X_1', Y_1', Z_1', W_1' are the tangential coördinates of the locus of T_1. In fact, it follows from (80), (85), (86), (87) and (88) that

$$\sum \xi_1 X_1' + \omega_1 W_1' = 0,$$

$$\sum \xi_1 \frac{\partial X_1'}{\partial v} + \omega_1 \frac{\partial W_1'}{\partial v} = 0. \tag{93}$$

Moreover, the condition

$$\sum \xi_1 \frac{\partial X_1'}{\partial u} + \omega_1 \frac{\partial W_1'}{\partial u} = 0$$

follows from

$$\sum \frac{\partial x}{\partial v}\frac{\partial X}{\partial u} + \frac{\partial w}{\partial v}\frac{\partial W}{\partial u} = 0,$$

which is a consequence of (85) and their derivatives.

In like manner it can be shown that the tangential coördinates X_1'', Y_1'', Z_1'', W_1'' of the locus of T_2 are of the form

$$X_1'' = \int X\left(\frac{\partial \mu_1}{\partial u} - \beta\mu_1\right) du + \mu_1\left(\frac{\partial X}{\partial v} + \alpha X\right) dv. \tag{94}$$

From (5), (86), (89) and (90) we have

$$\theta_1 = \sum X_1 x + W_1 w,$$

$$\lambda_1 = \sum x_1 X + w_1 W. \tag{95}$$

When the values of λ_1 and θ_1 from (89) and (86) are substituted in (4) and (83), the resulting equations are reducible to

$$\sigma_1 = \int \left(\sum X_1' \frac{\partial x_1'}{\partial u} + W_1' \frac{\partial w_1'}{\partial u}\right) du + \left(\sum X_1' \frac{\partial x_1'}{\partial v} + W_1' \frac{\partial w_1'}{\partial v}\right) dv,$$

$$\bar{\sigma}_1 = \int \left(\sum x_1' \frac{\partial X_1'}{\partial u} + w_1' \frac{\partial W_1'}{\partial u}\right) du + \left(\sum x_1' \frac{\partial X_1'}{\partial v} + w_1' \frac{\partial W_1'}{\partial v}\right) dv.$$

Hence by a suitable choice of the additive constants of integration we have

$$\sigma_1 + \bar{\sigma}_1 = \sum X_1' x_1' + W_1' w_1'. \tag{96}$$

In consequence of these results we have from (5) and (90)

$$\sum X_1 x_1 + W_1 w_1 = 0.$$

Since also

$$\sum \xi_2 X_1' + \omega_2 W_1' = 0,$$

it follows from (9), (90), (91) and (93) that

$$\sum X_1 \frac{\partial x_1}{\partial u} + W_1 \frac{\partial w_1}{\partial u} = 0, \qquad \sum X_1 \frac{\partial x_1}{\partial v} + W_1 \frac{\partial w_1}{\partial v} = 0.$$

Hence equations (9) and (84) define the same transformation T of S.

By making use of the results of § 5, we can obtain the equations of the theorem of permutability of transformations T from the standpoint of tangential coördinates. The functions λ_{12} and λ_{21} must satisfy

$$(97) \qquad \frac{\partial \lambda_{ij}}{\partial u} = \bar{\tau}_i \frac{\partial}{\partial u}\left(\frac{\lambda_j}{\lambda_i}\right), \qquad \frac{\partial \lambda_{ij}}{\partial v} = -\bar{\sigma}_1 \frac{\partial}{\partial v}\left(\frac{\lambda_j}{\lambda_i}\right) \quad \left(\begin{matrix} i = 1, 2, \\ j = 1, 2, \end{matrix} \; i \neq j\right).$$

The functions $\bar{\sigma}_{12}, \bar{\sigma}_{21}, \bar{\tau}_{12}, \bar{\tau}_{21}$ are given by

$$(98) \qquad \begin{aligned} \bar{\tau}_1 \bar{\tau}_{12} &= \bar{\tau}_2 \bar{\tau}_{21} = \frac{\lambda_2 \lambda_{21} \bar{\tau}_1}{\lambda_1} + \frac{\lambda_1 \lambda_{12} \bar{\tau}_2}{\lambda_2} - \lambda_{12} \lambda_{21}, \\ \bar{\sigma}_1 \bar{\sigma}_{12} &= \bar{\sigma}_2 \bar{\sigma}_{21} = -\left(\frac{\lambda_2 \lambda_{21} \bar{\sigma}_1}{\lambda_1} + \frac{\lambda_1 \lambda_{12} \bar{\sigma}_2}{\lambda_2} + \lambda_{12} \lambda_{21}\right), \end{aligned}$$

and the tangential coördinates of S_{12}, namely $X_{12}, Y_{12}, Z_{12}, W_{12}$, are of the form

$$(99) \qquad \lambda_1 \lambda_{12} X_{12} = \lambda_2 \lambda_{21} X_1 + \lambda_1 \lambda_{12} X_2 - \lambda_{12} \lambda_{21} X.$$

If equations similar to (95) are to be satisfied, we must have

$$(100) \qquad \begin{aligned} \lambda_{12} &= \sum X_1 x_{12} + W_1 w_{12} = \sum X_1 x_2 + W_1 w_2 - \theta_{21}, \\ \lambda_{21} &= \sum X_2 x_{12} + W_2 w_{12} = \sum X_2 x_1 + W_2 w_1 - \theta_{12}. \end{aligned}$$

When these equations are differentiated, we find that the resulting equations are satisfied in virtue of the preceding formulas. Hence we may take λ_{12} and λ_{21} as given by (100).

Equations similar to (5) and (90) are

$$x'_{12} = x_{12} + \frac{\sigma_{12}}{\theta_{12}} x_1 = x_2 - \frac{\theta_{21}}{\theta_1} x - \frac{x_1}{\sigma_1}\left(\frac{\theta_1 \sigma_2}{\theta_2} + \theta_{21}\right),$$

$$X'_{12} = X_{12} + \frac{\bar{\sigma}_{12}}{\lambda_{12}} X_1 = X_2 - \frac{\lambda_{21}}{\lambda_1} X - \frac{X_1}{\bar{\sigma}_1}\left(\frac{\lambda_1 \bar{\sigma}_2}{\lambda_2} + \lambda_{21}\right).$$

From these equations, (98) and (100) we obtain

$$\sum x'_{12} X'_{12} + w'_{12} W'_{12} = \sigma_{12} + \bar{\sigma}_{12}.$$

Consequently when λ_{12} and λ_{21} have the values (100), the expressions (99) are the tangential coördinates of S_{12} whose point coördinates are given by (45). From the form of (99) we are led at once to

THEOREM 8. *When S, S_1, S_2, S_{12} form a quatern for transformations T, four corresponding tangent planes meet in a point.*

During the remainder of this section we assume that the point coördinates are cartesian and that X, Y, Z are the direction-cosines of the normal to S. Consequently $-W$ is the distance from the origin to the tangent plane.

From (58) it follows that x_1' as it appears in (89) is equal to $x\sigma_1' + x^{(1)}$ and $w_1' = \sigma_1'$. Hence (89) may be replaced by

$$\lambda_1 = Xx^{(1)} + Yy^{(1)} + Zz^{(1)}. \tag{101}$$

Consequently λ_1 is the distance from the origin to the tangent plane to $S^{(1)}$.

We note that λ_1 and μ_1 determine a transformation of $S^{(1)}$. If X_1, Y_1, Z_1 are the direction-cosines of the normal to the transform $S_1^{(1)}$, and $-W_1^{(1)}$ the distance from the origin to the tangent plane to $S_1^{(1)}$, equations (84) are replaced by

$$\frac{\partial}{\partial u}\left(\frac{X_1}{W_1^{(1)}}\right) = -\bar{\tau}_1\frac{\partial}{\partial u}\left(\frac{X}{\lambda_1}\right), \qquad \frac{\partial}{\partial v}\left(\frac{X_1}{W_1^{(1)}}\right) = \bar{\sigma}_1\frac{\partial}{\partial v}\left(\frac{X}{\lambda_1}\right). \tag{102}$$

But X_1, Y_1, Z_1 are the direction-cosines of the normal to S_1 also. Moreover, the function W_1 is given by

$$\frac{\partial}{\partial u}\left(\frac{W_1}{W_1^{(1)}}\right) = -\bar{\tau}_1\frac{\partial}{\partial u}\left(\frac{W}{\lambda_1}\right), \qquad \frac{\partial}{\partial v}\left(\frac{W_1}{W_1^{(1)}}\right) = \bar{\sigma}_1\frac{\partial}{\partial v}\left(\frac{W}{\lambda_1}\right). \tag{103}$$

11. Transformations Ω of conjugate systems with equal tangential invariants

When equation (81) has equal invariants, it can be written

$$\frac{\partial^2\lambda_1}{\partial u\partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial\lambda_1}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial\lambda_1}{\partial v} + \gamma\lambda_1 = 0.$$

Analogously to the results of § 9 we have that $\mu_1 = 2\rho\lambda_1$ is a solution of the adjoint of this equation. For this value we have

$$\bar{\sigma}_1 = \tau_1 = \lambda_1^2\,\rho,$$

so that the tangential equation of the transform is

$$\frac{\partial^2\lambda_1'}{\partial u\partial v} - \frac{\partial}{\partial v}\log \sqrt{\rho}\lambda_1\frac{\partial\lambda_1'}{\partial u} - \frac{\partial}{\partial u}\log \sqrt{\rho}\lambda_1\frac{\partial\lambda_1'}{\partial v} + \gamma_1\lambda_1' = 0,$$

which also has equal invariants.

If we put

$$\vartheta_1 = \lambda_1\sqrt{\rho}, \qquad \vartheta_1' = \lambda_1'/\sqrt{\rho}\lambda_1,$$

these equations are equivalent to

$$\frac{\partial^2\vartheta_1}{\partial u\partial v} = \left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial u\partial v} - \gamma\right)\vartheta_1, \qquad \frac{\partial^2\vartheta_1'}{\partial u\partial v} = \left(\sqrt{\rho}\lambda_1\frac{\partial^2}{\partial u\partial v}\frac{1}{\sqrt{\rho}\lambda_1} - \gamma_1\right)\vartheta_1'.$$

These equations are satisfied respectively by the functions

$$\nu_1 = \sqrt{\rho}X, \qquad \nu_2 = \sqrt{\rho}Y, \qquad \nu_3 = \sqrt{\rho}Z;$$
$$\bar{\nu}_1 = X_1/\sqrt{\rho}\lambda_1, \qquad \bar{\nu}_2 = Y_1/\sqrt{\rho}\lambda_1, \qquad \bar{\nu}_3 = Z_1/\sqrt{\rho}\lambda_1.$$

In terms of these functions equations (84) are reducible to

$$(84') \qquad \begin{aligned} \frac{\partial}{\partial u}(\bar{\nu}_i\,\vartheta_1) &= \left(\vartheta_1\frac{\partial \nu_i}{\partial u} - \nu_i\frac{\partial \vartheta_1}{\partial u}\right), \\ \frac{\partial}{\partial v}(\bar{\nu}_i\,\vartheta_1) &= -\left(\vartheta_1\frac{\partial \nu_i}{\partial v} - \nu_i\frac{\partial \vartheta_1}{\partial v}\right) \end{aligned} \qquad (i = 1, 2, 3).$$

Since the tangential equation of S has equal invariants, there exists a surface Σ with this spherical representation of its asymptotic lines. Its point coördinates ξ, η, ζ are given by the Lelieuvre formulas

$$\frac{\partial \xi}{\partial u} = \nu_2\frac{\partial \nu_3}{\partial u} - \nu_3\frac{\partial \nu_2}{\partial u}, \qquad \frac{\partial \xi}{\partial v} = -\nu_2\frac{\partial \nu_3}{\partial v} + \nu_3\frac{\partial \nu_2}{\partial v}.*$$

Similar equations in the functions $\bar{\nu}_i$ give the point coördinates of a surface Σ_1 with the same spherical representation of its asymptotic lines as the parametric system on S_1. Moreover, equations (84′) are the condition that Σ and Σ_1 be the focal surfaces of a W-congruence.†

From the theory of W-congruences it follows that, if X, Y, Z and X_1, Y_1, Z_1 are the direction-cosines (not merely direction-parameters), of the normals to Σ and Σ_1 respectively, then

$$\nu_1 = \sqrt{\rho}X, \qquad \bar{\nu}_1 = \sqrt{\rho_1}\,X_1,$$

where $-1/\rho^2$ and $-1/\rho_1^2$ are the gaussian curvatures of Σ and Σ_1 respectively. Hence (84′) may be written

$$\frac{\partial}{\partial u}(\sqrt{\rho\rho_1}\,\lambda_1 X_1) = \rho\lambda_1^2\frac{\partial}{\partial u}\left(\frac{X}{\lambda_1}\right), \qquad \frac{\partial}{\partial v}(\sqrt{\rho\rho_1}\,\lambda_1 X_1) = -\rho\lambda_1^2\frac{\partial}{\partial v}\left(\frac{X}{\lambda_1}\right).$$

Comparing these equations with (102), we have

$$W_1^{(1)} = \frac{1}{\sqrt{\rho\rho_1}\,\lambda_1},$$

and equations (103) become

$$\frac{\partial}{\partial u}(\sqrt{\rho\rho_1}\,\lambda_1 W_1) = -\rho\lambda_1^2\frac{\partial}{\partial u}\left(\frac{W}{\lambda_1}\right), \qquad \frac{\partial}{\partial v}(\sqrt{\rho\rho_1}\,\lambda_1 W_1) = \rho\lambda_1^2\frac{\partial}{\partial v}\left(\frac{W}{\lambda_1}\right).$$

But these are the equations of transformations Ω of conjugate systems with equal tangential invariants previously found by the author.‡ Moreover, the

* E., p. 193. A reference of this sort is to the author's *Differential Geometry.*

† E., p. 419.

‡ M_2, §§ 1, 3.

special surfaces $S^{(1)}$ and $S_1^{(1)}$, treated in § 7, were used in the discussion of the transformations Ω.*

12. The extended theorem of permutability

In this section we extend the theorem of permutability so as to involve a group of eight surfaces. Let S_1, S_2, S_3 be three transforms of S by means of functions θ_i, ϕ_i for $i = 1, 2, 3$ respectively. Applying the theorem of permutability to the three pairs of these surfaces, we get three new surfaces S_{12}, S_{23}, S_{31}. We recall that $S_{ij} = S_{ji}$. Since S_{12} and S_{13} are transforms of S_1, there exists a family of surfaces S', for each of which S_1, S_{12}, S_{13}, S' form a quatern. It is our purpose to show that one of these surfaces S' is such that S_2, S_{12}, S_{23}, S' form a quatern; and likewise S_3, S_{13}, S_{23}, S'.

We denote by θ'_{12}, w'_{12}, ϕ'_{12} the functions transforming S_{12} into S'. The equations analogous to the first of (35) and (36) are

$$\begin{aligned} \theta_{12}\,\theta'_{12}\,w'_{12} &= w_{13}(\theta_{13}\,\theta'_{13} + \theta_{12}\,\theta'_{12} - \theta'_{12}\,\theta'_{13}), \\ \theta_{12}\,\theta'_{12}\,w'_{12}\,x' &= w_{13}(\theta_{13}\,\theta'_{13}\,x_{12} + \theta_{12}\,\theta'_{12}\,x_{13} - \theta'_{13}\,\theta'_{12}\,x_1). \end{aligned} \tag{104}$$

In consequence of (36) and an analogous expression for x_{13} the second of (104) is reducible to

$$\begin{aligned} \theta_{12}\,\theta'_{12}\,w'_{12}\,x' = x_1\Big(& w_{12}\,\theta'_{12}\,w_{13}\,\theta'_{13} \\ & - \frac{\theta_2\,\theta_{21}}{\theta_1\,\theta_{12}} w_{13}\,\theta'_{13}\,w_2\,\theta_{13} - \frac{\theta_3\,\theta_{31}}{\theta_1\,\theta_{13}} w_{12}\,\theta'_{12}\,w_3\,\theta_{12}\Big) \\ & - x_2\,w_{13}\,\theta'_{13}\,w_2\,\theta_{13} - x_3\,w_{12}\,\theta'_{12}\,w_3\,\theta_{12} \\ & + \frac{x}{\theta_1}(w_2\,\theta_{21}\,w_{13}\,\theta'_{13}\,\theta_{13} + w_3\,\theta_{31}\,w_{12}\,\theta'_{12}\,\theta_{12}). \end{aligned} \tag{105}$$

In deriving these equations, we looked upon S' as a transform of S_{12} which in turn is a transform of S_1. Looking upon S_{12} as a transform of S_2 we get the analogous equations

$$\begin{aligned} \theta_{21}\,\theta'_{21}\,w'_{21} &= w_{23}(\theta_{21}\,\theta'_{21} + \theta_{23}\,\theta'_{23} - \theta'_{21}\,\theta'_{23}), \\ \theta_{21}\,\theta'_{21}\,w'_{21}\,x' &= w_{23}(\theta_{21}\,\theta'_{21}\,x_{12} + \theta_{23}\,\theta'_{23}\,x_{23} - \theta'_{21}\,\theta'_{23}\,x_2). \end{aligned}$$

The latter equation is reducible to

$$\begin{aligned} \theta_{21}\,\theta'_{21}\,w'_{21}\,x' = & - x_1\,w_{23}\,\theta'_{23}\,w_1\,\theta_{23} - x_3\,w_{21}\,\theta'_{21}\,w_3\,\theta_{21} \\ & + x_2\Big(w_{21}\,\theta'_{21}\,w_{23}\,\theta'_{23} - w_{21}\,\theta'_{21}\,w_3\,\theta_{21}\frac{\theta_3\,\theta_{32}}{\theta_2\,\theta_{23}} \\ & \qquad - w_{23}\,\theta'_{23}\,w_1\,\theta_{23}\frac{\theta_1\,\theta_{12}}{\theta_2\,\theta_{21}}\Big) \\ & + \frac{x}{\theta_2}(w_3\,\theta_{32}\,w_{21}\,\theta'_{21}\,\theta_{21} + w_1\,\theta_{12}\,w_{23}\,\theta'_{23}\,\theta_{23}). \end{aligned} \tag{106}$$

* M_3, § 6.

From their definition it follows that θ'_{21} and w'_{21} are the same functions as θ'_{12} and w'_{12} respectively. Making use of this fact, we eliminate x' from (105) and (106). In the reduction we note that from (35) we have

$$w_1\, w_{12}\, \theta_1\, \theta_{12} = w_2\, w_{21}\, \theta_2\, \theta_{21}.$$

The resulting equation is of the form

$$Ax_1 + Bx_2 + Cx = 0,$$

where A, B, and C are determinate functions. These functions must equal zero, since equations similar to the above hold also in the y's and z's. This gives the three equations

$$\begin{aligned} \theta_1\, w_{12}\, \theta'_{12}\, w_{13}\, \theta'_{13} - \theta_2\, w_{13}\, \theta'_{13}\, w_2\, \theta_{21}\frac{\theta_{13}}{\theta_{12}} - \theta_3\, w_{12}\, \theta'_{12}\, w_3\, \theta_{31}\frac{\theta_{12}}{\theta_{13}} \\ + \theta_2\, w_2\, \theta_{23}\, w_{23}\, \theta'_{23} = 0, \\ \theta_2\, w_{21}\, \theta'_{21}\, w_{23}\, \theta'_{23} - \theta_3\, w_{21}\, \theta'_{21}\, w_3\, \theta_{32}\frac{\theta_{21}}{\theta_{23}} - \theta_1\, w_{23}\, \theta'_{23}\, w_1\, \theta_{12}\frac{\theta_{23}}{\theta_{21}} \\ + \theta_1\, w_1\, \theta_{13}\, w_{13}\, \theta'_{13} = 0, \\ w_{13}\, \theta'_{13}\, w_1\, \theta_{13}\, w_2\, \theta_{21} + w_{12}\, \theta'_{12}\, w_1\, \theta_{12}\, w_3\, \theta_{31} - w_{21}\, \theta'_{21}\, w_2\, \theta_{21}\, w_3\, \theta_{32} \\ - w_{23}\, \theta'_{23}\, w_1\, \theta_{12}\, w_2\, \theta_{23} = 0. \end{aligned} \tag{107}$$

It is readily found that these equations are equivalent to the three

$$\theta_i\, w_{ij}\, \theta'_{ij} = \theta_j\, w_j\, \theta_{ji}\frac{\theta_{ik}}{\theta_{ij}} + \theta_i\, w_j\, \theta_{jk} - \theta_k\, w_j\, \theta_{ji} \qquad \left(\begin{matrix} i = 1, 2,\ j = 1, 2,\ k = 1, 2 \\ i \neq j \neq k \end{matrix}\right). \tag{108}$$

When these values are substituted in (105), the result is reducible to

$$\begin{aligned} \Phi x' = {} & x_1(\theta_2\, \theta_{31}\, \theta_{23} + \theta_3\, \theta_{21}\, \theta_{32} - \theta_1\, \theta_{23}\, \theta_{32}) \\ & + x_2(\theta_1\, \theta_{13}\, \theta_{32} + \theta_3\, \theta_{12}\, \theta_{31} - \theta_2\, \theta_{31}\, \theta_{13}) \\ & + x_3(\theta_1\, \theta_{23}\, \theta_{12} + \theta_2\, \theta_{21}\, \theta_{13} - \theta_3\, \theta_{21}\, \theta_{12}) \\ & + x(\theta_{12}\, \theta_{23}\, \theta_{31} + \theta_{21}\, \theta_{13}\, \theta_{32}), \end{aligned} \tag{109}$$

where

$$\begin{aligned} \Phi &= \frac{\theta_1\, \theta_{12}\, w_{12}\, \theta'_{12}\, w'_{12}}{w_2\, w_3} \\ &= \theta_1(\theta_{13}\, \theta_{32} + \theta_{12}\, \theta_{23} - \theta_{32}\, \theta_{23}) + \theta_2(\theta_{31}\, \theta_{23} + \theta_{21}\, \theta_{13} - \theta_{13}\, \theta_{31}) \\ &\quad + \theta_3(\theta_{12}\, \theta_{31} + \theta_{32}\, \theta_{21} - \theta_{12}\, \theta_{21}) + \theta_{13}\, \theta_{32}\, \theta_{21} + \theta_{12}\, \theta_{23}\, \theta_{31}. \end{aligned} \tag{110}$$

When we proceed with S' as a transform of S_{13} or S_{23}, we arrive at the same result, which evidently is of symmetric form.

It remains for us to show that the functions θ'_{ij} as given by (108) satisfy equations analogous to (26), namely

$$(111)\quad \frac{\partial}{\partial u}(w_{ij}\theta'_{ij}) = \tau_{ij}\frac{\partial}{\partial u}\left(\frac{\theta_{ik}}{\theta_{ij}}\right), \qquad \frac{\partial}{\partial v}(w_{ij}\theta'_{ij}) = -\sigma_{ij}\frac{\partial}{\partial v}\left(\frac{\theta_{ik}}{\theta_{ij}}\right).$$

We know that this is true, since (108) for $i = 1$, $j = 2$, $k = 3$ follows from (36) when x_{12}, x_1, x_2, x are replaced by $\theta'_{12}, \theta_{13}, \theta_{23}, \theta_3$ respectively; and these results are general. Hence we have the extended theorem of permutability:

THEOREM 9. *If S, S_1, S_2, S_{12}; S, S_2, S_3, S_{23}; S, S_3, S_1, S_{13} are three quaterns of surfaces, a surface S' can be found, without quadrature, such that S_1, S_{12}, S_{13}, S'; S_2, S_{12}, S_{23}, S'; S_3, S_{13}, S_{23}, S' are quaterns.*

13. Relations between transformations T and radial transformations

If ω is a solution of equation (18), the surface $\bar{S}$ whose coördinates $\bar{x}, \bar{y}, \bar{z}$, are given by

$$(112)\qquad \bar{x} = \frac{x}{\omega}, \qquad \bar{y} = \frac{y}{\omega}, \qquad \bar{z} = \frac{z}{\omega},$$

is referred to a conjugate system. In fact, the point equation of $\bar{S}$ is

$$(113)\qquad \frac{\partial^2 \bar{\theta}}{\partial u \partial v} + \left(a + \frac{\partial \log \omega}{\partial v}\right)\frac{\partial \bar{\theta}}{\partial u} + \left(b + \frac{\partial \log \omega}{\partial u}\right)\frac{\partial \bar{\theta}}{\partial v} = 0.$$

We say that $\bar{S}$ is obtained from S by a *radial* transformation, since the line joining any pair of corresponding points on S and $\bar{S}$ passes through a point—the origin.

If θ_1 is a solution of (18), then $\bar{\theta}_1 = \theta_1/\omega$ is a solution of (113). Also it can be shown that if ϕ_1 is a solution of the adjoint equation of (18), then $\bar{\phi}_1 = \phi_1\,\omega$ is a solution of the adjoint of (113).

We consider the transformation T of $\bar{S}$ by means of these functions $\bar{\theta}_1$ and $\bar{\phi}_1$. If $\bar{\tau}_1$ and $\bar{\sigma}_1$ denote functions analogous to τ_1 and σ_1, it is readily found that to within additive constants we have

$$\bar{\tau}_1 = \tau_1, \qquad \bar{\sigma}_1 = \sigma_1.$$

Assuming these values, we have from equations (20) and the analogous ones,

$$\frac{\partial}{\partial u}(\bar{x}_1\,\bar{w}_1) = \bar{\tau}_1\frac{\partial}{\partial u}\left(\frac{\bar{x}}{\bar{\theta}_1}\right), \qquad \frac{\partial}{\partial v}(\bar{x}_1\,\bar{w}_1) = -\bar{\sigma}_1\frac{\partial}{\partial v}\left(\frac{\bar{x}}{\bar{\theta}_1}\right),$$

by integration

$$(114)\qquad \bar{x}_1\,\bar{w}_1 = x_1\,w_1$$

to within an additive constant. Also $\bar{w}_1$ is given by

$$(115)\quad \frac{\partial \bar{w}_1}{\partial u} = \bar{\tau}_1\frac{\partial}{\partial u}\left(\frac{1}{\bar{\theta}_1}\right) = \tau_1\frac{\partial}{\partial u}\left(\frac{\omega}{\theta_1}\right), \qquad \frac{\partial \bar{w}_1}{\partial v} = -\sigma_1\frac{\partial}{\partial v}\left(\frac{\omega}{\theta_1}\right).$$

Evidently there exists a function ω_1 defined by

$$(116)\qquad \frac{\partial}{\partial u}(\omega_1 w_1) = \tau_1 \frac{\partial}{\partial u}\left(\frac{\omega}{\theta_1}\right), \qquad \frac{\partial}{\partial v}(\omega_1 w_1) = -\sigma_1 \frac{\partial}{\partial v}\left(\frac{\omega}{\theta_1}\right),$$

which equations are similar to (20). Comparing (115) and (116), we note that ω_1 can be chosen so that

$$(117)\qquad \omega_1 w_1 = \bar{w}_1,$$

and consequently we have from (114)

$$(118)\qquad \bar{x}_1 = \frac{x_1}{\omega_1}, \qquad \bar{y}_1 = \frac{y_1}{\omega_1}, \qquad \bar{z}_1 = \frac{z_1}{\omega_1}.$$

Hence we have

THEOREM 10. *If two surfaces S and $\bar{S}$ are in the relation of a radial transformation and S_1 is a T transform of S, a surface $\bar{S}_1$ can be found by a quadrature which is a radial transform of S_1 and a T transform of $\bar{S}$.*

When in particular $\omega = \theta_1$, then $\bar{\theta}_1 = 1$, and consequently $\bar{S}$ and $\bar{S}_1$ are parallel. Now in all generality we take

$$\bar{w}_1 = -1, \qquad \omega_1 = -1/w_1.$$

Therefore, we have

THEOREM 11. *A transformation T is equivalent to the combination of an axial, a parallel and an axial transformations.*

Consider now a general quatern of surfaces S, S_1, S_2, S_{12}. From (116), (26) and analogous equations it follows that the functions

$$\omega = \theta_2, \qquad \omega_1 = \theta_{12}, \qquad \omega_2 = \frac{1}{w_2},$$

determine axial transformations of S, S_1 and S_2 respectively, into $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$. The equations determining the axial transform of S_{12} as of the pair S_1 and S_{12}, are

$$\frac{\partial}{\partial u}(\omega_{12} w_{12}) = \tau_{12} \frac{\partial}{\partial u}\left(\frac{\omega_1}{\theta_{12}}\right), \qquad \frac{\partial}{\partial v}(\omega_{12} w_{12}) = -\sigma_{12} \frac{\partial}{\partial v}\left(\frac{\omega_1}{\theta_{12}}\right).$$

In consequence of the preceding equations we may take $\omega_{12} = 1/w_{12}$.

In order to show that the same function ω_{12} determines the axial transform of S_{12}, as of the pair S_2 and S_{12}, the following equations must be satisfied:

$$(119)\qquad \frac{\partial}{\partial u}\left(\frac{w_{21}}{w_{12}}\right) = \tau_{21} \frac{\partial}{\partial u}\left(\frac{1}{w_2 \theta_{21}}\right), \qquad \frac{\partial}{\partial v}\left(\frac{w_{21}}{w_{12}}\right) = -\sigma_{21} \frac{\partial}{\partial v}\left(\frac{1}{w_2 \theta_{21}}\right).$$

When the values of w_{12}, w_{21}, σ_{21} and τ_{21}, as given by (35) and analogous equations are substituted, it is found that (119) are satisfied. Hence the surfaces

It remains for us to show that the functions θ'_{ij} as given by (108) satisfy equations analogous to (26), namely

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We know that this is true, since (108) for $i = 1$, $j = 2$, $k = 3$ follows from (36) when x_{12}, x_1, x_2, x are replaced by θ'_{12}, θ_{13}, θ_{23}, θ_3 respectively; and these results are general. Hence we have the extended theorem of permutability:

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Assuming these values, we have from equations (20) and the analogous ones,

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by integration

$$(114)\qquad \bar{x}_1\bar{w}_1 = x_1 w_1$$

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$$(115)\qquad \frac{\partial\bar{w}_1}{\partial u} = \bar{\tau}_1\frac{\partial}{\partial u}\left(\frac{1}{\bar{\theta}_1}\right) = \tau_1\frac{\partial}{\partial u}\left(\frac{\omega}{\theta_1}\right), \qquad \frac{\partial\bar{w}_1}{\partial v} = -\sigma_1\frac{\partial}{\partial v}\left(\frac{\omega}{\theta_1}\right).$$

Evidently there exists a function ω_1 defined by

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which equations are similar to (20). Comparing (115) and (116), we note that ω_1 can be chosen so that

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$$\omega = \theta_2, \qquad \omega_1 = \theta_{12}, \qquad \omega_2 = \frac{1}{w_2},$$

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$$\frac{\partial}{\partial u}(\omega_{12} w_{12}) = \tau_{12} \frac{\partial}{\partial u}\left(\frac{\omega_1}{\theta_{12}}\right), \qquad \frac{\partial}{\partial v}(\omega_{12} w_{12}) = -\sigma_{12} \frac{\partial}{\partial v}\left(\frac{\omega_1}{\theta_{12}}\right).$$

In consequence of the preceding equations we may take $\omega_{12} = 1/w_{12}$.

In order to show that the same function ω_{12} determines the axial transform of S_{12}, as of the pair S_2 and S_{12}, the following equations must be satisfied:

$$(119) \qquad \frac{\partial}{\partial u}\left(\frac{w_{21}}{w_{12}}\right) = \tau_{21} \frac{\partial}{\partial u}\left(\frac{1}{w_2 \theta_{21}}\right), \qquad \frac{\partial}{\partial v}\left(\frac{w_{21}}{w_{12}}\right) = -\sigma_{21} \frac{\partial}{\partial v}\left(\frac{1}{w_2 \theta_{21}}\right).$$

When the values of w_{12}, w_{21}, σ_{21} and τ_{21}, as given by (35) and analogous equations are substituted, it is found that (119) are satisfied. Hence the surfaces

$\bar{S}, \bar{S}_1, \bar{S}_2, \bar{S}_{12}$ form a quatern. Their coördinates are of the form

$$\bar{x} = \frac{x}{\theta_2}, \qquad \bar{x}_1 = \frac{x_1}{\theta_{12}}, \qquad \bar{x}_2 = x_2 w_2, \qquad \bar{x}_{12} = x_{12} w_{12},$$

the transformation functions being

$$\begin{array}{lll} \bar{\theta}_1 = \dfrac{\theta_1}{\theta_2}, & \phi_1 = \theta_2 \phi_1, & \bar{w}_1 = w_1 \theta_{12}; \\ \bar{\theta}_2 = 1, & \bar{\phi}_2 = \theta_2 \phi_2, & \bar{w}_2 = 1; \\ \bar{\theta}_{12} = 1, & \bar{\phi}_{12} = \theta_{12} \phi_{12}, & \bar{w}_{12} = 1; \\ \bar{\theta}_{21} = w_2 \theta_{21}, & \bar{\phi}_{21} = \dfrac{\phi_{21}}{w_2}, & \bar{w}_{21} = \dfrac{w_{21}}{w_{12}}. \end{array}$$

These functions satisfy equations analogous to (35).

In a similar manner we get a second quatern by using θ_1 for the axial transformation of S. Hence we have

THEOREM 12. *When a quatern of surfaces is known, two other quaterns each containing two pairs of parallel surfaces can be found without quadrature, and these surfaces are axial transforms of the surfaces of the given quatern.*

As a matter of fact axial transformations can be looked upon as special types of transformations T. For if we take

$$\tau_2 = -\sigma_2 = 1, \qquad \theta_2 = \omega - 1, \qquad w_2 = \frac{1}{\theta_2} + 1 = \frac{\omega}{\omega - 1},$$

equations of the form (19) and (20) with subscripts 2 instead of 1 are integrable in the form

$$x_2 = \frac{x}{\omega}, \qquad y_2 = \frac{y}{\omega}, \qquad z_2 = \frac{z}{\omega}.$$

Let us apply the theorem of permutability to the case in which S_2 is given as above. One solution of (37) is

$$\theta_{21} = \frac{\theta_1}{\omega_2 \theta_2} = \frac{\theta_1}{\omega}.$$

We find also

$$\tau_{12} = -\sigma_{12} = 1, \qquad \sigma_{21} = \sigma_1, \qquad \tau_{21} = \tau_1,$$

$$w_{12}\theta_{12} = 1 + \theta_{12}, \qquad w_{21} = w_1(1 + \theta_{12}).$$

Hence if we put $\omega_1 = \theta_{12} + 1$, equation (36) in this case reduces to (118). Consequently the theorem of permutability is equally true when an axial transformation is used. It is readily shown also that theorem 12 can be established by means of the generalized results of § 11.

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